A note on stratified domination and 2-rainbow domination in graphs

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In this paper relations between stratified domination and 2-rainbow domination in graphs are investigated. And we conjectured that these two parameters are equal or 2-rainbow domination number is greater than stratified domination number by one.

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1. Introduction

In this paper, we follow the notation of [1]. Specifically, let $G=(V,E)$ be a connected graph with vertex set V and edge set E. $\delta(G)$ and $\Delta(G)$ denote the minimum degree and maximum degree of G, respectively. For any vertex $v\in V$, the open neighborhood of v is the set $N(v)=\{u\in V \mid uv\in E\}$ and the closed neighborhood is the set $N[v]=N(v)\cup\{v\}$. For a set $S\subseteq V$ the open neighborhood is $N(S)=\bigcup_{v\in S}N(v)$ and the closed neighborhood is $N[S]=N(S)\cup S$. A set $S\subseteq V$ is a dominating set of G if $N(S)=V$. The domination number of G, denoted by $\gamma(G)$ is the minimum cardinality of dominating set. The concept of domination in graphs, with its many variations, is now well studied in graph theory. A thorough study of domination appears in [2]. The 2-rainbow domination defined as follows. Let f be a function that assigns to each vertex a set of colors chosen from the set $A=\{1,2\}$ that is $f:V(G)\rightarrow\{\varnothing,\{1\},\{2\},A\}$. If for each vertex $v\in V(G)$ such that $f(v)\neq\varnothing$ or $f(v)=\varnothing$ when $\bigcup_{u\in N(v)}f(u)=A$. Then f is called a 2-rainbow dominating function of G. which we denote by $\gamma_{r2}(G)$. The weight, $w(f)$, of a function f is defined as $w(f)=\sum_{v\in V}\left|f(v)\right|$. Given a graph G, the minimum weight of a 2-rainbow dominating function is called the 2-rainbow domination number of G, which we denote by $\gamma_{r2}(G)$. For detailed discussion of 2-rainbow domination see [3,4,5,6,7].

A graph $G=(V,E)$ together with a fixed partition of its vertex set V into nonempty subsets is called a stratified graph. If the partition is $V=\{V_1, V_2\}$ then G is a 2-stratified graph and the sets V_1 and V_2 are called the strata or sometimes the color classes of G. We ordinarily color the vertices of V_1 red and the vertices of V_2 blue. In [8], Rashidi studied a number of problems involving stratified graphs; while distance in stratified graphs was investigated in [9,10,11]. The study combining stratification and domination in graphs was started by Chartrand et al. [12]. Let F be a 2-stratified graph with one fixed blue vertex v specified. We say that F is rooted at the blue vertex v. An F-coloring of a graph G is defined in [12] to be a red blue coloring of the vertices of G such that every blue vertex v of G belongs to a copy of F (not necessarily induced in G) rooted at v. The F-domination number $\gamma_F\left(G\right)$ of G is the minimum number of red vertices of G in an F-coloring of G. An F-coloring of G that colors $\gamma_F\left(G\right)$ vertices red is called a γ_F-coloring of \$G\$. The set of red vertices in a γ_F-coloring is called a γ_F-set. If G has order n and G has no copy of F, then certainly $\gamma_F\left(G\right)=n$. For a nice survey on this topic we encourage the reader to consult [13]. The concepts of stratification and domination in graphs may be extended in a number of ways. In [12], the following extension is considered. Let $\mathsf{F}=\{F_1,F_2,...,F_n\}$ where F_i, $1\leq i\leq n$,2-stratified graph rooted at some blue vertex v. The F-coloring of a graph G is a red-blue coloring of the vertices of G such that every blue vertex v of G belongs to a copy of F_i rooted at v for every value of i.

The $\mathcal{F}$-domination number $\gamma_{\mathcal{F}}(G)$ of G is the minimum number of red vertices of G in an $\mathcal{F}$-coloring of G. In [13], this definition was extended as follows. Let $\mathcal{F}=\{F_1,\ldots,F_n\}$ where F_i, $1\le i\le n$, 2-stratified graph rooted at some blue vertex v. The $\mathcal{F}^*$-coloring of a graph G is a red-blue coloring of the vertices of G such that every blue vertex v of G belongs to a copy of F_i rooted at v for at least one value of i. The $\mathcal{F}^*$-domination number $\gamma_{\mathcal{F}^*}(G)$ of G is the minimum number of red vertices of G in an $\mathcal{F}^*$-coloring of G. In this paper as $\mathcal{F}$ to be 2-stratified P_3 in which two vertices are colored red we investigate $\mathcal{F}^*$-domination number for the some special graph classes and compare these results with the 2-rainbow domination numbers.

2. Main results

Let F be a 2-stratified P_3 rooted at a blue vertex v. The five possible choices for the graph F are shown in Fig. 1. (The red vertices in Fig. 1 are darkened.)

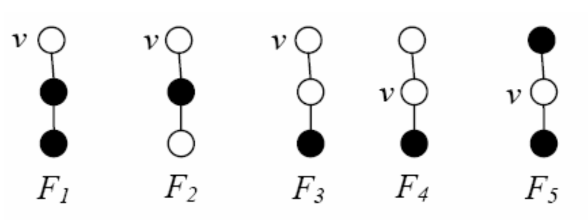

Fig. 1. The five 2-stratified graphs P_3

Proposition 1. Let $\mathcal{F}=\{F_1,F_5\}$,

(i) $\gamma_{\mathcal{F}^*}(P_n)=\left\lfloor \frac{n}{2}\right\rfloor+1 \quad (n\equiv 1,2,3 \quad mod 4)$

(ii) $\gamma_{\mathcal{F}^*}(P_n)=\left\lfloor \frac{n}{2}\right\rfloor \quad (n\equiv 0 \quad mod 4)$

Proof. (i) Let us consider a path in which the vertex set is $\{v_1,\ldots,v_n\}$. If n is odd, $v_1,v_3,\ldots,v_{n-2},v_n$ vertices can be colored red and this gives an $\mathcal{F}^*$-coloring of P_n. If $n\equiv 2(mod 4)$ then $v_1,v_3,\ldots,v_{n-1}$ and v_n vertices can be colored red and this gives an $\mathcal{F}^*$-coloring of P_n. Hence $\gamma_{\mathcal{F}^*}(P_n)\le\left\lfloor \frac{n}{2}\right\rfloor+1$. Let us accept that $\gamma_{\mathcal{F}^*}(P_n)=\left\lfloor \frac{n}{2}\right\rfloor$. If n is odd, the number of blue vertices are greater than red vertices by one. This is a contradiction. If $n\equiv 2(mod 4)$ then the number of red vertices are equal to the number of blue vertices. In this case let k be a positive integer and $n=4k+2$. Then the vertices $v_1,v_3,\ldots,v_{4k+2}$ are red and the vertices $v_2,v_4,\ldots,v_{4k+1}$ are blue. But if any $\mathcal{F}^*$-coloring of P_n which rooted at the blue vertex v_{4k+2}, the vertices v_{4k+1} and v_{4k} must be colored red. This is a contradiction. Thus $\gamma_{\mathcal{F}^*}(P_n)\ge\left\lfloor \frac{n}{2}\right\rfloor+1$.

(ii) In this case the vertices $v_2,v_3,v_6,v_7,\ldots,v_{n-2},v_{n-1}$ can be colored red and and this gives an $\mathcal{F}^*$-coloring of P_n. Hence $\gamma_{\mathcal{F}^*}(P_n)\le\left\lfloor \frac{n}{2}\right\rfloor$. Let us accept that $\gamma_{\mathcal{F}^*}(P_n)=\left\lfloor \frac{n}{2}\right\rfloor-1$. In this situation the number of blue vertices are greater than red vertices by one. This is a contradiction. Thus $\gamma_{\mathcal{F}^*}(P_n)\ge\left\lfloor \frac{n}{2}\right\rfloor$.

For 2-rainbow domination number of paths, Brešar and Šumenjak [3] showed the following observation.

Observation 1 ([3]). $\gamma_{r2}(P_n)=\left\lfloor \frac{n}{2}\right\rfloor+1$.

Let now G be a graph of P_n. With Proposition 1, the following corollary is obtained.

Corollary 1. Let $\mathcal{F}=\{F_1,F_5\}$,

(i) $\gamma_{r2}(P_n)=\gamma_{\mathcal{F}^*}(P_n) \quad (n\equiv 1,2,3 \quad mod 4)$

(ii) $\gamma_{r2}(P_n)=\gamma_{\mathcal{F}^*}(P_n)+1 \quad (n\equiv 0 \quad mod 4)$

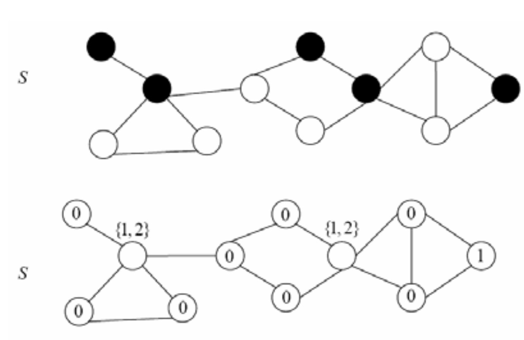

Fig. 2. F^*-domination and 2-rainbow domination in the grap.

Proposition 2. Let $\mathrm{F}=\{F_1,F_5\}$,

(i) $\gamma_{\mathrm{F}^*}\left(C_n\right)=\left\lfloor \frac{n}{2}\right\rfloor+1$, n is odd.

(ii) $\gamma_{\mathrm{F}^*}\left(C_n\right)=\left\lfloor \frac{n}{2}\right\rfloor$, n is even.

Proof. (i) Let us consider C_n with the vertex set $\{v_1,v_2,\ldots,v_n\}$. If the vertices $v_1,v_3,\ldots,v_{n-2},v_{n-1}$ are colored red, then this gives an F^*-coloring of C_n. Hence $\gamma_{\mathrm{F}^*}\left(C_n\right)\leq\left\lfloor \frac{n}{2}\right\rfloor+1$. Let us accept that $\gamma_{\mathrm{F}^*}\left(C_n\right)=\left\lfloor \frac{n}{2}\right\rfloor$. In this case the number of blue vertices are greater than red vertices. This is a contradiction. Thus $\gamma_{\mathrm{F}^*}\left(C_n\right)\geq\left\lfloor \frac{n}{2}\right\rfloor+1$.

(ii) If the vertices $v_1,v_3,\ldots,v_{n-1}$ are colored red, then this gives an F^*-coloring of C_n.

Hence $\gamma_{\mathrm{F}^*}\left(C_n\right)\leq\left\lfloor \frac{n}{2}\right\rfloor$. Let us accept that $\gamma_{\mathrm{F}^*}\left(C_n\right)=\left\lfloor \frac{n}{2}\right\rfloor-1$. In this case the number of blue vertices are greater than red vertices. This is a contradiction. Thus $\gamma_{\mathrm{F}^*}\left(C_n\right)\geq\left\lfloor \frac{n}{2}\right\rfloor$.

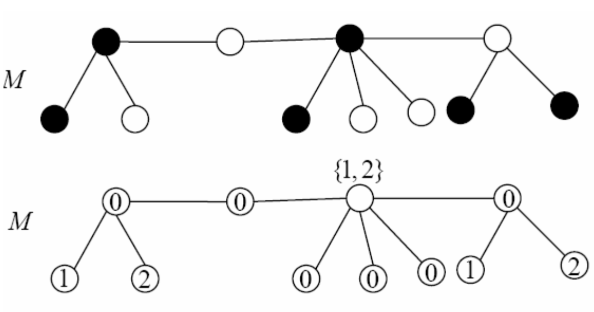

Fig. 3: F^*-domination and 2-rainbow domination in the graph M.

For 2-rainbow domination number of cycles, Brešar and Šumenjak [3] proved the following proposition.

Proposition 3 ([3]). For $n\geq 3$, $\gamma_{r2}\left(C_n\right)=\left\lfloor \frac{n}{2}\right\rfloor+\left\lceil \frac{n}{4}\right\rceil-\left\lfloor \frac{n}{4}\right\rfloor$.

Note that $\left\lfloor \frac{n}{2}\right\rfloor+\left\lceil \frac{n}{4}\right\rceil-\left\lfloor \frac{n}{4}\right\rfloor=\left\lfloor \frac{n}{2}\right\rfloor+1$, $(n\equiv 1,2,3 \ mod 4)$ and $\left\lfloor \frac{n}{2}\right\rfloor+\left\lceil \frac{n}{4}\right\rceil-\left\lfloor \frac{n}{4}\right\rfloor=\left\lfloor \frac{n}{2}\right\rfloor$, $(n\equiv 0 \ mod 4)$. Let now G be a graph of C_n. With Proposition 2, the following corollary is obtained.

Corollary 2. Let $\mathrm{F}=\{F_1,F_5\}$,

(i) $\gamma_{r2}\left(C_n\right)=\gamma_{\mathrm{F}^*}\left(C_n\right),(n\equiv 0,1,3 \ mod 4)$.

(ii) $\gamma_{r2}(C_n)=\gamma_{\mathrm{F}^*}(C_n)+1$, $(n\equiv 2 \ mod 4)$.

Proposition 4. Let $\mathrm{F}=\{F_1,F_5\}$ and let $S_{1,n-1}$ be a star, $\gamma_{\mathrm{F}^*}(S_{1,n-1})=2$.

Proof. Let v_0 be the central vertex of $S_{1,n-1}$ and let the other vertices are $v_1,\ldots,v_{n-1}$. If the vertex v_0 and any vertex of $S_{1,n-1}$ are colored red then this gives an F^*-coloring of $S_{1,n-1}$. Hence $\gamma_{\mathrm{F}^*}\left(S_{1,n-1}\right)\leq 2$. Let us accept that $\gamma_{\mathrm{F}^*}\left(S_{1,n-1}\right)=1$. Any F^*-coloring of a graph must contains at least two red vertex. Thus $\gamma_{\mathrm{F}^*}(S_{1,n-1})\geq 2$. The proof is completed.

In the same way the F^*-domination number of K_n is $\gamma_{\mathrm{F}^*}\left(K_n\right)=2$.

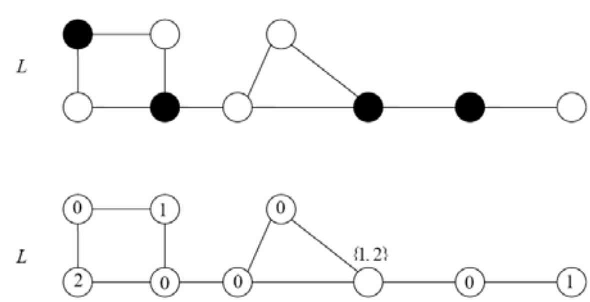

Fig. 4. F^*-domination and 2-rainbow domination in the graph L.

For 2-rainbow domination number of stars and complete graphs, $\gamma_{r2}\left(S_{1,n-1}\right)=\gamma_{r2}\left(K_n\right)=2$ can be written. Then we can state the following corollary.

Corollary 3. Let $\mathrm{F}=\{F_1,F_5\}$ then;

(i) $\gamma_{r2}\left(S_{1,n-1}\right)=\gamma_{\mathrm{F}^*}\left(S_{1,n-1}\right)$.

(ii) $\gamma_{r2}\left(K_n\right)=\gamma_{\mathrm{F}^*}\left(K_n\right)$.

Randomly chosen three graphs and their F^*-domination numbers and 2-rainbow domination numbers are shown in the figures 2,3 and 4.

In the light of these observations and results in this paper, we believe that the following conjecture holds.

Conjecture 1. Let $\mathsf{F}=\{F_1,F_5\}$ and G be a connected graph, $\gamma_{r2}(G)=\gamma_{\mathsf{F}^*}(G)$ or $\gamma_{r2}(G)=\gamma_{\mathsf{F}^*}(G)+1$.

References

[1] B. Bollobás, Modern Graph Theory, 2nd ed., Springer-Verlag New York, Inc., 1998.
[2] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1998.
[3] B. Breŝar, T.K. Ŝumenjak, On the 2-rainbow domination in graphs, Discrete Appl. Math. **155**, 2394 (2007).
[4] T. Chunling, L. Xiaohui, Y. Yuanshenga, Discrete Appl. Math. **157**, 1932 (2009) .
[5] G. Xu, Discrete Appl. Math. **157**, 2570 (2009) .
[6] G. J. Chang, J. Wu, X. Zhu, Discrete Appl.Math. **157**, 8 (2010) .
[7] Y. Wu, H. Xing, , Appl. Math. Lett. **23**, 706 (2010) .
[8] R. Rashidi, The Theory and Applications of Stratified Graphs, Ph.D.Dissertation, Western Michigan University (1994).
[9] G. Chartrand, L. Eroh, R. Rashidi, M. Schultz, N.A. Sherwani, Congr. Numer. **107**, 81 (1995) .
[10] G. Chartrand, H. Gavlas, M.A. Henning, R. Rashidi, Stratidistance in stratified graphs, Math. Bohem. **122,** 337 (1997).
[11] G. Chartrand, L. Holley, R. Rashidi, N.A. Sherwani, Distance in stratified graphs, Czechoslovak Math. J. **50**, 35 (2000) .
[12] G. Chartrand, T.W. Haynes, M.A. Henning, P. Zhang, Stratification and domination in graphs, Discrete Math. **272**, 171 (2003).
[13] T. Haynes, M.A. Henning, P. Zhang, A survey of stratified domination in graphs, Discrete Math. **272**, 5806 (2009) .

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