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A three-dimensional magnetic field and electromagnetic force computation technique based on the Fast Multipole Method

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This paper presents an efficient implementation of the Fast Multipole Method for the computation of magnetic field and electromagnetic force. The total computational cost of our technique is proportional to $O(N\log N)$, where N is the total number of discretization points in the system. Sample numerical results are presented for the vector potential, magnetic field, and magnetic forces in systems of solenoids interacting magnetically.

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1. Introduction

The Fast Multipole Method (FMM) has been called one of the "ten most significant algorithms" in scientific computation discovered in the 20th century [1, 2]. The method allows the evaluation of the product between a dense matrix (having some particular structure) and a vector in $O(N\log N)$ operations, whereas direct multiplication requires $O(N^2)$ operations, where N is the dimension of the matrix. Although the FMM was initially developed for the computation of the electrostatic potential created by a large number of charges [3], lately it has found applications in a large number of areas ranging from chemistry and physics, to computer visions and finance [4-9]. In this paper we present for the first time an implementation of the FMM algorithm for the computation of magnetic fields and magnetic forces in electromagnetics. Our implementation is based on the discretization of the kernel of integrals in the Biot-Savart and Biot-Savart-Laplace laws and the efficient summation of the resulting "magnetic charges" by using the FMM algorithm.

The paper is organized as follows. Section II presents the basic idea of our technique for the computation of vector potentials (Section 2.A), magnetic fields (Section 2.B), and forces (Section 2.C) in magnetic systems. Numerical results are presented in Section III and conclusions are drawn in Section IV.

2. Technical discussion

Consider the electromagnetic system presented in Fig. 1 in which current I is flowing through a wire Γ. We discretize the wire into N segments, whose ends are of coordinates (x_i, y_i, z_i), $i = 0.,..,N$. The length of each segment can be approximated by using finite differences:

$$\Delta \boldsymbol{l}_i = (x_i - x_{i-1})\,\boldsymbol{i} + (y_i - y_{i-1})\,\boldsymbol{j} + (z_i - z_{i-1})\,\boldsymbol{k}, \quad i = 1.,..,N \qquad (1)$$

while the centers are given by the vectors:

$$\boldsymbol{r}_i' = \frac{x_{i+1} + x_i}{2}\,\boldsymbol{i} + \frac{y_{i+1} + y_i}{2}\,\boldsymbol{j} + \frac{z_{i+1} + z_i}{2}\,\boldsymbol{k}\,. \qquad (2)$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the unit vectors in the x, y, and z directions, respectively.

2.1 Computation of the Magnetic Vector Potential

The α coordinate of the magnetic vector potential produced by current I can be calculated by using:

$$A_\alpha(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int_\Gamma \frac{dl_\alpha}{|\boldsymbol{r} - \boldsymbol{r}_i'|} \approx \frac{\mu_0 I}{4\pi}\sum_{i=1}^{N} \frac{\Delta l_{i,\alpha}}{|\boldsymbol{r} - \boldsymbol{r}_i'|}, \quad \alpha = x, y, z \qquad (3)$$

where μ_0 is the permeability of free space, $\Delta l_{i,\alpha}$ is the α coordinate of $\Delta \boldsymbol{l}_i$, and $\boldsymbol{r}$ represents the distance vector from the origin to the destination point. The summation in equation (3) can be computed efficiently by using the Fast Multipole Method (FMM) introduced in [3]. In the framework of the FMM the summation in (3) can be expressed as:

$$A_\alpha(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\sum_{i=1}^{N} \Delta l_{i,\alpha} N(\boldsymbol{r}_i')\Theta F(\boldsymbol{r}) = \hat{N}\Theta F(\boldsymbol{r}), \text{ if } r > r' \qquad (4)$$

$$A_\alpha(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\sum_{i=1}^{N} \Delta l_{i,\alpha} F(\boldsymbol{r}_i')\Theta N(\boldsymbol{r}) = \hat{F}\Theta N(\boldsymbol{r}), \text{ if } r < r', \qquad (5)$$

where:

$$\hat{N} = \frac{\mu_0 I}{4\pi}\sum_{i=1}^{N} \Delta l_{i,\alpha} N(\boldsymbol{r}_i') \quad \text{and} \quad \hat{F} = \frac{\mu_0 I}{4\pi}\sum_{i=1}^{N} \Delta l_{i,\alpha} F(\boldsymbol{r}_i')\,. \qquad (6)$$

In equations (4) and (5) N and F are the near field and far field tensors that can be expressed in terms of the associated Legendre polynomials $P_{lm}(\cos\theta)$ by using:

$$F_{lm}(\boldsymbol{r}) = \frac{(l-m)!P_{lm}(\cos\theta)e^{im\phi}}{r^{l+1}} \text{ and } N_{lm}(\vec{r}) = \frac{r^l P_{lm}(\cos\theta)e^{-im\phi}}{(l+m)!}. \quad (7)$$

In equations (4) and (5) Θ stands for the tensor contraction with respect to indices l and m: $A\Theta B = \sum_{l=0}^{l_{\max}} \sum_{m=-l}^{l} A_{lm}B_{lm}$, where $l_{\max}$ is the maximum order of Legendre polynomials used in the summation. In our simulations we consider terms up to $l_{\max} = 8$, which gives an error in the final results less then 0.1%. The far field and near field tensors are computed by using the algorithm presented in [4] and the magnetic vector potential is then determined by using equations (4) and (5). It should be noted that, once quantities $\hat{N}$ and $\hat{F}$ are computed, the computational overhead for the computation of $A_\alpha(\boldsymbol{r})$ is given by the summation of $2^{l_{\max}}$ terms, which is usually a much smaller number than the total number of discretization points N. Hence, the computational cost would be much higher if the integral in equation (3) were computed by using direct summation.

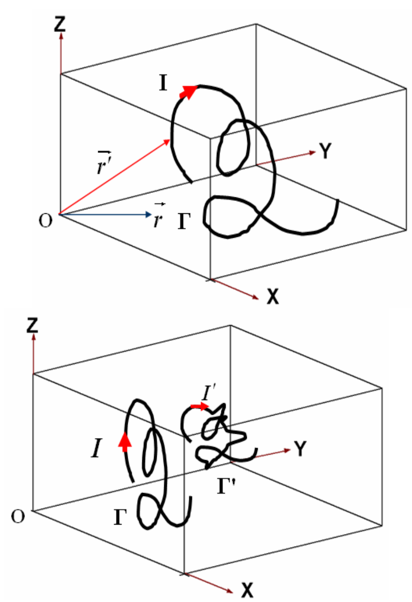

Fig. 1. (a) Magnetic system consisting of a wire carrying a current I. (b) Magnetic system consisting of two interacting wires that carry currents I and I'.

2.2 Computation of the Magnetic Field

The magnetic field created by the circuit in Fig. 1 cannot be computed by using the technique described in the previous section because the magnetic field cannot be written as a contraction of two far and near field tensors. Indeed, the magnetic field created by current I is given by $\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int_\Gamma \frac{d\boldsymbol{l}\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}_i'|^3}$, and the kernel $\frac{d\boldsymbol{l}\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}_i'|^3}$ cannot be expressed as the dot product of two vectors that depend on $\boldsymbol{r}$ and $\boldsymbol{r}'$. For this reason, it is easier to use equations (4)-(5) and to compute the magnetic field as $\boldsymbol{B} = \nabla\times\boldsymbol{A}$. The derivatives of the vector potential can be evaluated in terms of the derivatives of the far field and near filed tensors:

$$\frac{\partial N_{lm}}{\partial r} = \frac{l}{r}N_{lm},\ \frac{\partial N_{lm}}{\partial\phi} = -\frac{im}{r\sin(\theta)}N_{lm},$$
$$\frac{\partial N_{lm}}{\partial\theta} = -\frac{(l+m+1)e^{i\phi}}{2}N_{l,m+1} + \frac{(l-m+1)e^{-i\phi}}{2}N_{l,m-1} \quad (8)$$

$$\frac{\partial F_{lm}}{\partial r} = -\frac{l+1}{r}F_{lm},\ \frac{\partial F_{lm}}{\partial\phi} = -\frac{im}{r\sin(\theta)}F_{lm},$$
$$\frac{\partial F_{lm}}{\partial\theta} = -\frac{(l+m+1)e^{i\phi}}{2}F_{l,m+1} + \frac{(l-m+1)e^{-i\phi}}{2}F_{l,m-1}. \quad (9)$$

For instance, the derivative of $A_\alpha(\boldsymbol{r})$ with respect to r can be computed as: $\frac{\partial A_\alpha(\boldsymbol{r})}{\partial r} = \frac{\mu_0 I}{4\pi}\sum_{i=1}^{N}\sum_{l=0}^{l_{\max}}\sum_{-l}^{l}\Delta l_{i,\alpha}N_{lm}(\boldsymbol{r}_i')\frac{\partial F_{lm}(\boldsymbol{r})}{\partial r} = -\frac{\mu_0 I}{4\pi}\sum_{i=1}^{N}\sum_{l=0}^{l_{\max}}\sum_{-l}^{l}\frac{l+1}{r}\Delta l_{i,\alpha}N_{lm}(\boldsymbol{r}_i')F_{lm}(\boldsymbol{r})$ for $r > r'$ and as $\frac{\partial A_\alpha(\boldsymbol{r})}{\partial r} = \frac{\mu_0 I}{4\pi}\sum_{i=1}^{N}\sum_{l=0}^{l_{\max}}\sum_{-l}^{l}\frac{l}{r}\Delta l_{i,\alpha}F_{lm}(\boldsymbol{r}_i')N_{lm}(\boldsymbol{r})$ for $r < r'$. Similarly, one first can compute the derivatives of the vector potential with respect to θ and φ, and then evaluate $\boldsymbol{B}$.

2.3 Computation of electromagnetic forces

The magnetic force between two wires carrying currents I and I' can be calculated by using the Biot-Savart-Laplace law:

$$\boldsymbol{F} = \frac{I'\mu_0}{4\pi}\int_{\Gamma'} d\boldsymbol{l}'\times\boldsymbol{B}(\boldsymbol{r}') = \frac{II'\mu_0}{4\pi}\int_{\Gamma'} d\boldsymbol{l}'\times\int_{\Gamma+\Gamma'}\frac{[d\boldsymbol{l}\times(\boldsymbol{r}-\boldsymbol{r}')]}{|\boldsymbol{r}-\boldsymbol{r}_i'|^3}. \quad (10)$$

If the two integrals in the last equations are discretized in $N\times N$ mesh points, the total computational overhead for the evaluation of the magnetic force is proportional to $O(N^2)$, which might require long computational times on normal computers. However, by using the FMM algorithm the computational overhead for the calculation of the magnetic force is proportional to $O(N \log N)$.

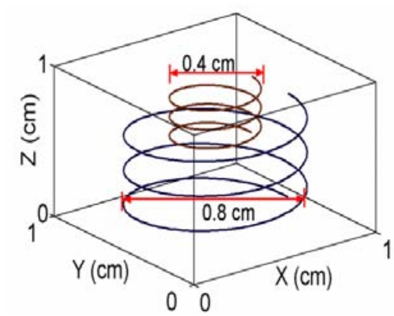

Fig. 2. Pair of solenoids used to test the numerical implementation of the FMM technique.

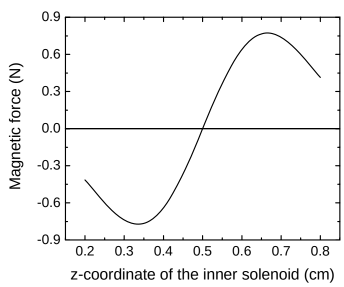

Fig. 3. The z-component of the force between two solenoids as a function of the position of the second solenoid.

3. Numerical results

The FMM algorithm presented in the previous section was numerically implemented and used to compute the vector potential, magnetic filed, and magnetic forces produced by the system of two solenoids shown in Fig. 2. The two solenoids have 8 turns each; the length and radius of the outer solenoid are 0.6 cm and 0.4 cm, respectively; the length and radius of the outer solenoid are 0.3 cm and 0.2 cm, respectively. The two solenoid were discretized in 800 segments (801 mesh points) and each of them carries a current of 1 A. The outer solenoid extends from $z = 0.2$ cm to $z = 0.8$ cm. In the simulations presented below, the position of the inner solenoid is varied along the $z-$axis. The magnetic force between the two solenoids is represented as a function of the $z-$coordinate of the center of the inner solenoid in Fig. 3. The force is symmetric with respect to the origin and is repulsive because the currents in the two solenoids have the same direction.

The vector potential and magnetic field created by the two solenoids at $z = 0.9$ cm are presented in figures 4(up) and 4(down) as functions of the (x, y) coordinates. Both the vector potential and the magnetic field present rotational symmetry with respect to the origin because of the symmetry of the problem.

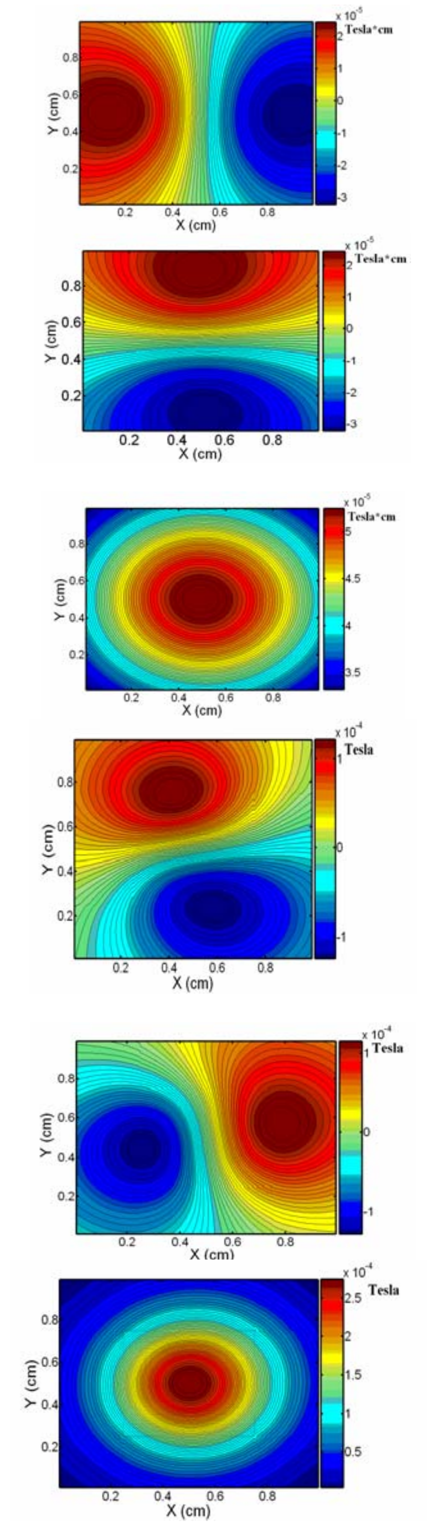

Fig. 4. The x, y, and z components of the vector potential (up) and magnetic filed (down) in plane $z = 0.9$ cm as a function of coordinates (x, y).

4. Conclusions

The Fast Multipole Method provides a very efficient algorithm for the numerical computation of vector potentials, magnetic fields, and magnetic forces in magnetic systems. The total number of operations in the evaluation of the magnetic fields and forces is decreased from $O(N^2)$ when direct summation is used, to $O(N \log N)$ in the case of FMM. Hence, the FMM algorithm can be used efficiently for the analysis of magnetic effects in wire systems.

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