Invited Lecture

An introduction to the Casimir effect in critical phenomena

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Confinement conditions imposed on a system whose correlation function decays as a power law induce a long-range force between the surfaces limiting the system. One can generally call this phenomenon the Casimir effect. Prominent examples are systems at critical points or systems with spontaneously broken global continuous symmetry that lead to massless modes: "spin waves" or Goldstone bosons. Constrained conditions imposed on such systems lead to the so called “statistical-mechanical” or “thermodynamic” Casimir effect, a condensed matter analogue of the “electromagnetic” or “quantum – mechanical” Casimir effect, due to the constrained zero-point vacuum fluctuations of the electromagnetic field, discussed for the first time by H. G. B. Casimir (1948). The present review is devoted to the theoretical and experimental results for the thermodynamic Casimir effect.

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1. Introduction

It is in some sense amazing that Casimir's discovered remarkable theoretical result was obtained as a by-product of totally applied industrial research, related to the stability of colloidal suspensions used to deposit films in the manufacture of lamps and cathode ray tubes. In the 1940s, Overbeek and Verwey at the Philips Laboratory studied the properties of suspensions of quartz powder and concluded that their experiment was not in agreement with the existing London-van der Waals theory. Overbeek suggested that this may be related to the finite speed of light. Casimir and Polder reconsidered the theory with retardation included. The result (with better agreement with experiment) was that the retardation caused the interaction to vary at large intermolecular distances as r^{-7} rather than r^{-6}. The presented theory was very elaborate, using a 4th order perturbation calculation in quantum electrodynamics. Looking for a simpler derivation, Casimir showed that a physical force can be generated by the change of zero point vacuum fluctuations due to the presence of two parallel conducting flat plates separated by a distance L. This force $F(L)$ per unit area, A, (i.e. pressure), at absolute zero temperature $T=0$, has the magnitude

$$F(L) = -\frac{\pi^2 \hbar c}{240 L^4} \qquad (1)$$

Only Planck’s constant $\hbar$ and the speed of light c enter Eq. (1). It is interesting to note that at distances of the order of 10 nm between the plates, the force may produce a pressure of the order of 1 atmosphere. For details of the background, experiments and applications of this issue, the reader can consult, e.g. the reviews [1,2] and refs. therein.

It was Bohr's idea that vacuum fluctuations could be at the origin of Polder and Casimir's results. One can consider the Casimir force as a striking macroscopic observation of the effects of vacuum fluctuations in quantum electrodynamics. Furthermore, it becomes clear that this effect is not limited to the realm of quantum electrodynamics. In terms of the behaviour of the two point correlation function of a system, one can take a more general point of view. Confinement conditions, imposed on a system whose correlation function decays as a *power* law in space, induce a long-range force between the surfaces limiting the system. One can generally call this phenomenon the Casimir effect. In other words, the effect is a phenomenon common to all systems characterised by fluctuating fields on which external boundary conditions are imposed.

Nowadays, one can distinguish, besides the electromagnetic Casimir effect, the Casimir effect in general quantum field theory, in cosmology, in particle physics, and in critical phenomena (it is also called the “statistical-mechanical” or “thermodynamic” Casimir effect) [2].

In the present review, we will focus our attention on the Casimir effect in critical phenomena: e.g. the forces between boundaries and layers due to long-range order at critical points and in phases with spontaneously broken global continuous symmetry that lead to massless modes: "spin waves" or Goldstone bosons.

The effect is related to the external constraints on a system undergoing a phase transition and its consequences, i.e. it is a typical finite-size effect. That is why the theory of critical phenomena in finite-size systems is an indispensable part of the theory of the “statistical-mechanical" Casimir effect. A list, focusing on various aspects of the statistical-mechanical" Casimir effect, can be found in [3-7] and references therein.

2. Finite-size systems and finite-size scaling

Experimental samples are always of finite size. They are characterized by the presence of surfaces and have certain shapes. The partition function of such a finite system is a polynomial of a finite degree, and thus never shows singularities. Critical points of the system in its literal sense are a result of the thermodynamic limit at which the volume has become infinite at constant particle density in the bulk. Exactly in the thermodynamic limit, critical points are characterized by singularities in the thermodynamic functions and by an infinite correlation length ξ. In a finite system, the correlation length ξ_L will not become infinite and the singularities in the free energy are replaced by rounded extrema located at some shifted position as compared to the position of the bulk singularities. To what extent an experimental sample can be regarded as of bulk or finite size depends on the value of the ratio $y = \xi/L$, where L is the effective value of the linear extensions of the system. For $y << 1$, any finite size effects will be invisible. If however $y = O(1)$, strong deviations from the bulk critical behaviour will be observed. Such behaviour has become known as finite - size critical behaviour. Since the beginning of the 1970s, when the basic ideas were given by Fisher, a comprehensive theory of finite-size critical behaviour has been developed, see [3, 4] and references therein.

In the following, it is convenient to distinguish the "regular" and "singular" parts for every thermodynamic function of a finite sample in the vicinity of the bulk critical point. A main characteristic feature of the bulk critical points is scaling. If a thermodynamic function depends on several variables, the singular part of this function depends only on a certain combination of these variables. In the case of two variables, e.g. L and ξ, one is scaled by a certain power of the other. If the singular part of a thermodynamic function depends on $y = \xi/L$, such a dependence is known as *standard finite-size scaling*. In experiments, if the properly scaled thermodynamic functions are plotted versus a properly scaled variable, finite-size scaling manifests itself as data collapse. Note that standard finite-size scaling reflects the *ordinary homogeneity* of the corresponding thermodynamic function of the finite system as a function of the two natural macroscopic lengths L and ξ [4].

Let us consider a statistical-mechanical system with slab geometry $L \times \infty^{d-1}$ under given boundary conditions τ imposed on the finite direction. If the system is at a critical point T_C or in a phase with broken continuous symmetry in its "bulk phase" (i.e. in the limit $L \to \infty$), it exhibits long-range correlations. These decay algebraically rather than exponentially fast, giving rise to a fluctuation-induced long-range force: the Casimir force in the critical phenomena. The existence of Casimir forces in finite-size critical systems was first anticipated by Fisher and. de Gennes [8]. At the bulk critical point T_C the total free energy density (per unit area and per $k_B T$) of a d-dimensional critical system in the form of a slab with thickness L, area A, and boundary conditions (a) and (b) at the opposite surfaces, has the asymptotic form

$$f_L^{(a,b)}(T_C) \cong L f_{bulk}(T_C) + f_{surf}^{(a)}(T_C) + f_{surf}^{(b)}(T_C) + L^{-(d-1)}\Delta^{(a,b)}(T_C) + \ldots \quad (2)$$

as $A \to \infty$, L >>1. Here $f_{surf}^{(a)}$ and $f_{surf}^{(b)}$ are the surface energy contributions and $\Delta^{(a,b)}$ is called the Casimir amplitude. The L dependence of the last term in Eq. (2) follows from the scale invariance of the free energy, as pointed out in [8]. The amplitude $\Delta^{(a,b)}$ is *universal*, depending on the bulk universality class and the universality classes of the boundary conditions [3, 4]. Let us recall that the theory classifying continuous phase transitions into universality classes is characterized by the spatial dimensions, the range of interactions and the symmetry of the order parameter.

The Casimir force between the slab faces is defined as

$$F^{(a,b)}(T,L) = -\frac{\partial f_{ex}^{(a,b)}(T,L)}{\partial L}, \quad (3)$$

where $f_{ex}^{(a,b)}$ (T,L) is the so called *excess free* energy, defined as

$$f_{ex}^{(a,b)}(T,L) = f_L^{(a,b)}(T) - L f_{bulk}(T)$$

Let us consider the *finite-size* part of the excess free energy

$$\delta f_L^{(a,b)}(T) \equiv f_L^{(a,b)}(T) - L f_{bulk}(T) - f_{surf}^{(a)}(T) - f_{surf}^{(b)}(T)$$

According to the finite-size scaling hypothesis for the "singular part" of $\delta f_L^{(a,b)}(T)$, we have [3,4]

$$\delta f_{ex,\sin g}^{(a,b)}(\mathrm{T,L}) = L^{(d-1)} X_{ex}^{(a,b)}\left(\mathrm{a}_t \mathrm{t\,L}^{1/\nu}\right), \quad (4)$$

where t=(T-T$_C$)/T$_C$ is the reduced temperature, a_t is a non-universal scaling factor, $X_{ex}^{(a,b)}$ is a *universal* (usually geometry dependent*) scaling function,* $X_{ex}^{(a,b)}(0) \equiv \Delta^{(a,B)}(T_C)$, and ν is the critical exponent of the correlation length. The non-singular, at T_c, part of $f_{ex}^{(a,b)}(T,L)$ is usually proportional to $O(L^{-d})$, and can be omitted. At the bulk critical point for the Casimir force, we obtain the result

$$F^{(a,b)}(T_C, L) = -(d-1)\frac{\Delta^{(a,b)}(T_C)}{L^d}. \tag{5}$$

The relation (4) is valid in the finite-size scaling region, when $t\, L^{1/\nu} = O(1)$. When the system leaves this region towards high temperatures, one usually expects a small excess free energy

$$f_{ex}^{(a,b)}(\mathrm{T,L}) = \mathrm{O}\left(\mathrm{e}^{-\mathrm{t\,L}}\right),$$

which means that the Casimir force equals zero. The situation is more complicated when $\mathrm{t\,L} \to -\infty$, i.e. below the critical temperature [3,4]. One expects, in phases with long–range correlations generated by the broken continuous symmetry, the value of $\Delta^{(a,b)}$ to be non-zero. In other words, Goldstone modes give rise to non-zero Casimir forces. In general, near T_c one has

$$F^{(a,b)}(T, L) = -L^{-d} X_{Cas}^{(a,b)}\left(a_t \mathrm{t\,L}^{\ 1/\nu}\right). \tag{6}$$

The *universal scaling functions* $X_{ex}^{(a,b)}(x)$ and $X_{Cas}^{(a,b)}(x)$ are the subject of the theory in the framework of different model calculations.

3. Model calculations

3.1 Perfect Bose-gas

Let us consider the model of a perfect Bose-gas (PBG). This allows a shortcut to the considered problem. It is known that the PBG undergoes a phase transition (the *Bose - Einstein condensation*) in the grand canonical ensemble, if the chemical potential $\mu \le 0$ equals the critical value $\mu_c = 0$. PBG is the simplest system showing spontaneously broken continuous symmetry with a long-distance power decrease of particle-particle correlations in the condensed phase. As was pointed out recently by Martin and Zagrebnov [9], the PBG may be a nice illustration of a critical system in which the Casimir phenomenon can take place. Being of a pedagogical interest, one can believe that this model underlies a number of its ultimate features. Let us note that the thermodynamics of a two-dimensional ideal boson film of thickness L and infinite lateral extent was studied by Barber and Fisher in the 1970s, using a different mathematical technique [10]. However in [10], the Casimir phenomenon has not been studied. At this point, below we will follow ref. [9]. The PBG grand-canonical thermodynamic potential (per unit area and per k_BT) $\varphi_L^{(a,b)}(T,\mu) \equiv \varphi_L^{(a,b)}$, in the case of slab geometry has the form

$$\varphi_L^{(a,b)} = -\frac{(1)}{(2\pi)^2}\int_{R^2} d^2\mathbf{q}\sum_{k_z}\ln\{1 - e^{\beta[\varepsilon(q)+\varepsilon(k_z)-\mu]}\}$$

where $\mathbf{q}$ is a two-dimensional wave vector in the (x, y) plane R^2, q = |**q**|, ε(q) = $\hbar^2(q_x^2+q_y^2)/2m$, $\varepsilon(k_z)$ = $\hbar^2 k_z^2/2m$, $\beta = (k_B \mathrm{T})^{-1}$ and μ is the chemical potential. The sum $\sum_{k_z}$ runs over the set depending on the boundary conditions (b.c.) in the finite z-direction. Let us consider the Dirichlet-Dirichlet b.c. (D) , then this sum runs over $k_z = \frac{\pi}{L} n_z$, n_z = 1, 2,… .The Bose condensation takes place only in the bulk limit, provided μ=0. For $\varphi_L^{(a,b)}(T,\mu=0)$ the following asymptotic result, with a correction decaying *faster than any power* of λ/L, is obtained [9]:

$$\varphi_L^{(D,D)}(\mathrm{T},0) = -Lp_{bulk}(T,0) + \varphi_{surf}^{(D)}(\mathrm{T},0) + \frac{1}{L^2}[\Delta^{(D)}(\mathrm{T}) + \mathrm{O}((\lambda/\mathrm{L})^M)], \tag{7}$$

for any $\mathrm{M} \ge 1$, where

$$\Delta^{(D)}(\mathrm{T}) = -\frac{\varsigma(3)}{8\pi} \approx -0.048, \tag{8}$$

$\lambda = \hbar\sqrt{\beta/m}$ denotes the thermal wavelength and $\varsigma(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta-function. Here p_{bulk} (T,0) is the standard Bose gas pressure (per $k_B T$) and $\varphi_{surf}^{(D)}(T,0)$ is its surface correction, both at $\mu = 0$. For Periodic (P) and Neumann (N) boundary conditions, the results are quite similar with $\Delta^{(P)} = 8\Delta^{(D)}$, $\Delta^{(N)} = \Delta^{(D)}$ and $\varphi_{surf}^{(N)}$(T,0)= $-\varphi_{surf}^{(D)}$(T,0), $\varphi_{surf}^{(P)} = 0$. Eq. (7) explicitly displays decomposition (2) for the grand canonical potential of the PBG. The result (8) is two times larger than that obtained in the free massless scalar field theory (see Eq. A.18.7 in [11]). It is well known that the problem of a phase transition in a Bose gas can be considered in terms of a classical complex field (see e.g.[12]). So, if one multiplies the scalar field result by two for the component of the complex field, both results would coincide. As we will see below, investigating the effects of this kinematically simpler two-component field allow one to study the problem.

Eq. (8) is relevant in the critical point, $\mu = 0$. A definite result can also be obtained outside the critical point. It was pointed out by Gambassi and Dietrich [13] that if $L/\lambda >> 1$,

$$\varphi_{ex}^{(D,D)}(L,u) = -\frac{1}{8\pi L^2}\sum_{n=1}^{\infty}\frac{1+2un}{n^3}e^{-2un}, \qquad (9)$$

where $u = (-2\beta\mu)^{\frac{1}{2}} L/\lambda$. The above formula can be expressed in terms of the polylogarithms $Li_p(p)$ defined by the series

$$Li_p(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^p}, \qquad (10)$$

with $|z| \le 1$ for $p \ge 2$ and $-1 \le z < 1$ for p=1. Polylogarithms appear to be a powerful tool in the thermodynamics of the perfect bulk Bose and Fermi gases [14, 15] and as well as in the theory of the finite-size scaling [4]. Thus, one can present Eq. (9) in the form

$$\varphi_{ex}^{(D,D)}(L,u) = -\frac{1}{8\pi L^2}[Li_3(e^{-2u}) + 2uLi_2(e^{-2u})] \qquad (11)$$

This form is convenient to generate small–u value corrections. If m = 0, 1, 2, … , the following $z \to 1$ expansion takes place [14], see also [15]

$$Li_{m+1}(z) = \sum_{r=0(r\neq m)}^{\infty}\frac{\varsigma(m+1-r)(\ln z)^r}{r!} + \frac{(-1)^{m+1}}{\Gamma(m+1)}\left(\ln\frac{1}{z}\right)^m\left(\ln\ln\frac{1}{z} - \psi(m+1) + \psi(1)\right), \qquad (12)$$

where ψ is the ln-derivative of the Γ-function. Using this formula, if in Eq. (11) one considers $e^{-2u} = z \to 1$, it is easy to obtain the first few *nonuniversal* corrections to Eq. (8). They are

$$\varphi_{ex}^{(D,D)}(L,u) = -\frac{1}{8\pi L^2}[\varsigma(3) + u^2(2\ln 2u - 1) + \ldots]. \qquad (13)$$

From Eqs. (3) and (11), for the Casimir force we obtain (c.f. [13])

$$F(L,T,\mu) = -\left[Li_3\left(e^{-2u}\right) + 2uLi_2\left(e^{-2u}\right) + 2u^2\ln\left(1-e^{-2u}\right)\right]\frac{1}{4\pi L^3}. \qquad (14)$$

From Eq. (14), at the transition point $u = 0$ ($\mu = \mu_c = 0$), for the Casimir force we obtain

$$F_{PBG} = -\frac{\varsigma(3)}{4\pi}\frac{1}{L^3}, \qquad (15)$$

and for the Casimir amplitude $\Delta^{(D)}$ one recovers Eq. (8).

Simple dimension analysis can explain the difference in power of L in Eqs. (1) and (15). The thermodynamic Casimir force (it is per unit area and per $k_B T$), as one can see, is classical in origin. This reflects the fact that the phase transition in the condensed state is governed by classical (or thermodynamic) fluctuations. This is in contrary to the electromagnetic Casimir force (per unit area), Eq. (1), which is quantum in origin and hence proportional to $\hbar c$ ($\hbar c$ has dimension of [energy times length]).

By the relation $u = (-2\beta\mu)^{\frac{1}{2}} L/\lambda, u \sim L/\xi_+$, where $\xi_+ \sim (\mu - \mu_c)^{-\frac{1}{2}}$, one can see that in the bulk regime $L/\xi_+ >> 1$ the Casimir force becomes exponentially small

$$F(L,T,\mu) = -2\left[\left(e^{-2u}\right) + 2u^2\left(e^{-2u}\right) + \ldots\right]\frac{1}{8\pi L^3}$$

and so invisible. Eq. (14) is one of the only few exact results (see also [16,17,18], where mean spherical and quantum spherical models are studied) concerning the Casimir phenomenon in critical systems. The above consideration can be performed also for Periodic and Neumann b.c.

3.2 More realistic model calculations

Liquid films are of particular experimental relevance to the Casimir phenomenon. Prominent candidates are films of ^4He close to the λ transition. Especially, in the past the Bose-Einstein condensation in PBG was considered as a sort of theory for the λ-transition (for comments on this issue see [10]). The qualitative estimate of [9] for liquid helium at $T = 2K$, where $\lambda \sim 4$ Å shows that a slab of thickness slightly larger than the interatomic distance already exhibits a Casimir effect. Indeed, to consider liquid helium as an ideal gas is illegal. The correct treatment requires more elaborate finite-size theory, for example in the framework of the Ginzburg-Landau model. It is well known that the Bose gas with weak interaction between the particles, near the transition point, can be described using the functional integration approach. The grand canonical partition function is a functional integral with weight $e^{-S[\phi]}$, where for the case of the Ginzburg-Landau model

$$S[\phi] = \int_0^L dz \int_{R^{d-1}} d^{d-1}x \{\frac{1}{2}[\nabla\phi(x,z)]^2 + \frac{1}{2}r\phi^2(x,z) + \frac{g}{4!}[\phi^2(x,z)]^2\}. \qquad (16)$$

One must also take into account the boundary conditions at the confining walls for the N component real field $\phi(x,z)$. In the case of a Bose gas, we have $N=2$, $r=-2m\mu/\hbar^2$, $g=48\pi a\hbar^4/\lambda^2$ and a is the scattering length. If $g=0$, Eq. (16) describes the exactly solvable Gaussian model (defined only for $r>0$). The Gaussian universal scaling functions for different boundary conditions are calculated by Krech and Dietrich [19]. The result for the scaling function, for the Dirichlet-Dirichlet b.c. $\phi(x,0)=\phi(x,L)=0$, is (in notation of ref. [19]):

$$\Theta^{(1)}_{+0,0}(y_+)= -2^{-d}\pi^{-d/2}N\frac{2\pi^{1/2}y_+^d}{\Gamma\left(\frac{d+1}{2}\right)}\int_1^\infty\frac{\left(t^2-1\right)^{(d-1)/2}}{\exp(2ty_+)-1}dt \tag{17}$$

If $d=3$, instead of Eq. (17) we have

$$\Theta^{(1)}_{+0,0}(y_+)= -\frac{N}{16\pi}[Li_3(e^{-2y_+})+2y_+Li_2(e^{-2y_+})]. \tag{18}$$

If $N=2$ and $y_+\equiv L/\xi_+$, from Eq. (18) one recovers the result (11), i.e. the PBG and the two component Gaussian model share the same universality class, as was mentioned above (see also [11]). It is important that this statement is valid under the condition $L>>\lambda$.

Strictly speaking, Eq. (17) is obtained above the bulk critical temperature, as an approximation to one-loop order. This approximation becomes exact (up to logarithmic corrections) in the upper critical dimension $d=4$. A further more refined consideration of the problem in the framework of the Ginzburg-Landau model is due to the field-theoretical analysis in $d=4-\varepsilon$ dimensions $(\varepsilon<<1)$. It allows universal quantities, like the scaling function at $d=3$, to be computed by a perturbation theory around the upper critical dimension $d_c=4$, so far up to first order in $\varepsilon=4-d$ [19,3]. The corresponding formula for the Casimir amplitude for the Dirichlet-Dirichlet b.c. is

$$\frac{\Delta_{0,0}}{N}=-\frac{\pi^2}{1440}\{1+\varepsilon[\ln\left(2\pi^{1/2}\right) +\frac{\gamma-1}{2}-\frac{\varsigma'(4)}{\varsigma(4)}-\frac{5}{4}\frac{N+2}{N+8}]\}. \tag{19}$$

Here, $\gamma\approx 0.577216$ denotes Euler's constant, $\varsigma'(4)\approx -0.068911$ and $\varsigma(4)=\pi^4/90$. The question of how trustworthy is result (19) for a three dimensional system, by setting $\varepsilon=1$, is quite difficult. The problem manifests itself in the case of Periodic b.c. Having in mind that we know the exact result in the limit $N\to\infty$ [16], contrary to the common expectation, the $O(\varepsilon)$ results for Δ^P do not approach, but deviate from, the exact one as N grows (see Fig.12.8 of [4]). Note that Monte-Carlo result for the Ising model $\Delta^P=-0.1526\pm 0.0010$ [20] is surprisingly close to the exact value $\Delta^P(N\to\infty)=-2\varsigma(3)/(5\pi)\approx -0.1530$ [16]. Thus, one may conclude that the first order in ε results have an incorrect $N-$dependent behavior, being too crude approximation in order to capture the way in which the exact $N\to\infty$ result is approached [4]. Especially for the case of Periodic b.c. recently Diehl, et al. [21] found that the ε-expansion is ill-defined beyond two-loop order, because of infrared singularities. The presented revised theory yields well-defined small-ε expansion involving fractional powers and logarithms of ε. The estimated values of Δ^P are: -0.1967 (for N=1), -0.2147 (for N=2) and -0.2311 (for N=3) [21], i.e. again we have a deviation from the exact $N\to\infty$ result -0.1530 as N grows. Although, in the case of Dirichlet-Dirichlet b.c., the ε-expansion has no infrared complications [21], further work is necessary to answer the question of "how reliable the ε-expansion is in the case of slab geometry in order to obtain $d=3$ results? ".

From the results of [19] one can see that the Casimir effect is an order of magnitude larger for Periodic and Antiperiodic b.c. In case of equal boundary conditions, the Casimir force is attractive. Unequal boundary conditions (Antiperiodic and Dirichlet-Neumann) lead to a positive Casimir amplitude, which means a repulsive Casimir force between the confining plates.

4. Experimental confirmations

While micro and nanoscale experiments precisely verify the original electromagnetic Casimir force, the experimental characterization of the thermodynamic Casimir force has been scarce and, often, ambiguous. The most reliable experimental testing of the thermodynamic Casimir effect is related to measurements on thin ^4He films at and near the superfluid/normal transition $T_c=T_\lambda=2.1768$ K. This is due to the nearly-ideal, impurity-free nature of liquid ^4He and its low-sensitivity to gravitational rounding errors [22, 23]. Because the superfluid order parameter vanishes at both film interfaces the Dirichlet-Dirichlet boundary conditions seem to be relevant [24]. This causes an attractive Casimir force, as follows from the theoretical considerations. If this force appears, it must produce near T_λ a temperature dependent change in the equilibrium film thickness.

Let us consider a surface of a substrate placed at a height h of a reservoir of liquid ^4He that is in coexistence with its vapor. A thin liquid film with thickness d is formed on the substrate. The film thickness d can be determined by the force balance equation among the gravitational, van der Waals and Casimir forces (see, e.g.[23,25])

$$mgh = \frac{\gamma_0}{d^3}\left(1+\frac{d}{d_{1/2}}\right)^{-1} + \frac{k_B T_\lambda V}{d^3} X_{Cas}^{(D,D)}(d/\xi)$$

Here, g is the gravitational acceleration, m is the atomic weight of helium, γ_0 and $d_{1/2}$ are specific interpolation parameters that characterize the van der Waals interaction of the liquid with the substrate, $V = 45.81\,\text{Å}^3/atom$ is the specific volume of liquid ^4He, and $X_{Cas}^{(D,D)}$ is the dimensionless Casimir force scaling function. Since the Casimir force is attractive, it favors a thinner film.

Thinning of the superfluid films was experimentally observed and studied in [22, 23]. The ^4He films were formed on the surfaces of polished Cu capacitor plates, set in a cell containing liquid ^4He at the bottom [22]. The force balance equation and the experimental data were utilized in order to obtain the Casimir force scaling function. A number of obstacles arise in such experiments related to minimization of surface roughness, precise control of the cleanness of the surfaces, precise control of the temperature etc. For example, the roughness of the Cu surface changes the effective areas of the Cu plates and makes the accurate determination of the film thickness impossible. Recently, there have been more precise capacitance measurements [23], where films of 238, 285, and 340 Å thickness adsorbed on $N-$doped silicon substrates with roughness ≈ 8Å were studied. In the region below T_λ where the effect is greatest, the scaling function $X_{Cas}^{(D,D)}(x)$, deduced from the thinning of these three films, collapses onto a single universal curve, attaining a $\min X_{Cas}^{(D,D)}(x) = -1.30 \pm 0.03$ at the value of the scaling variable $x = tL^{1/\nu} = -9.7 \pm 0.8\text{Å}^{1/\nu}$. The collapse confirms the finite-size scaling origin in the film thickness. Also, the presence down to $2.13K$ of the Goldstone/surface fluctuation force makes the superfluid film ~ 2Å thinner than the normal film [23].

One might ask about the correlation of the theoretical estimations with the experimental results. From a theoretical point of view, the situation is again quite complicated. The existing renormalization group calculations [19, 3] are valid only for $T \geq T_\lambda$. They are in good agreement with experiment only at the transition temperature [23, 25]. However, the regime $T < T_\lambda$ is more interesting, since there the effect is the greatest [23]. This regime is apparently far more difficult for a theoretical description and the existing theory is still under development [25, 26, 27]. Deep in the superfluid state, coming from the Goldstone modes in the bulk of the film, the following result for the Casimir force (per unit area and per $k_B T$)

$$F_{bulk} = -\frac{\varsigma(3)}{8\pi}\frac{1}{L^3},$$

has been obtained [28]. If only these modes caused the thinning of the film, the film would be roughly the same at the critical point and beneath the transition, which is not the experimental situation.

To resolve the discrepancy with the experimental observations, Zandi et al. [25] added the effect of surface fluctuations, which also act as a source of a Casimir force. In ref. [25], it was shown that surface fluctuations lead to an additional force

$$F_{surf} = -\frac{7}{4}\frac{\varsigma(3)}{8\pi}\frac{1}{L^3}$$

nearly twice as large as the bulk one. As pointed out in [23] the experimentally obtained thinning of the film is consistent with the following asymptotic, *low-temperature* value of the Casimir amplitude $\Delta_{\exp}^{^4He} \approx -0.30 \pm 0.10$, which is marginally larger than the theoretical result $\Delta_{theor}^{^4He} \approx -0.15$ [25], due to both the Goldstone modes and surface fluctuations.

As a result, one may conclude that the finite-size scaling theory and the thermodynamic Casimir effect are unambiguously confirmed by the experiment on two-dimensional liquid ^4He films of thickness d. However an accurate comparison between theory and experiment requires further precise measurements and more reliable theoretical estimations.

5. Applications and further investigations

One of the most important applications of the *electromagnetic* Casimir force is in nanotechnology. Characteristic sizes of microdevices and microstructures are of micrometres to nanometres, i.e. a distance scale in which the Casmir and van der Waals forces become dominant. The study of the Casimir phenomenon in superfluid systems at the moment seems to be totally fundamental research.

Liquid-crystal films provide an additional class of systems that are characterized by large thermal fluctuations, in conjunction with confining geometries. A prominent example is nematic liquid crystals in the nematically ordered phase, where fluctuations of the nematic director in a thin film geometry lead to long-range fluctuation induced forces. In liquid crystals, different critical regimes are easily accessible in comparison with superfluids, and what is important, occur at room temperature. This makes easier the experimental realization and the possible applications of the Casimir phenomenon. For literature on the subject, see [3, 4, 7, 29].

Some promising investigations seem to be related to quantum critical phenomena [30]. Quantum phase transitions occur at zero temperature as a function of some nonthermal parameter such as composition or pressure, and are driven by quantum fluctuations. In the context of the Casimir phenomenon, an exactly solvable quantum model was considered in [17]. In addition to being of interest for various low-temperature effects, quantum phase transitions are important because they are believed to play a crucial role in quantum information science. The famous Einstein, Podolsky and Rosen arguments [31] demonstrated that under some circumstances the quantum theory exhibits strange correlations, known as entanglement. These highly valuable circumstances may arise naturally in *finite-size systems* upon approaching a *quantum critical point*. Recently, it has been realized that entanglement can be utilized as a resource in quantum computing. One may speculate, therefore, about the role of the Casimir phenomenon in relation to this issue. Being an intrinsic feature of quantum phase transitions, entanglement inspired a whole rapidly expanding field of research on the frontier of the theory of phase transitions and quantum information theory [32].

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