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Higher-order variational analysis of finitely clad optical waveguides

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A higher-order variational method is applied to determine the normalized propagation constant in a finitely clad slab waveguide. We obtain closer results to the exact values in comparison with the first-order perturbation of the infinite clad waveguide, first–order perturbation of the coreless structure and the two–sided perturbation methods. The accuracy of the method depends on how many trial eigenfunctions are used, on the near and far cut-off regimes, on the ratio *a*/λ and also on the refractive index differences.

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1. Introduction

The principle of the first-order perturbation of the infinite clad waveguide, first–order perturbation of the coreless structure and the two–sided perturbation methods is to replace a complex waveguide structure by its sub–structures whose solutions are much simpler to obtain [1].

The scalar-wave equation for a finitely clad slab waveguide is given by

$$\frac{d^2\psi(x)}{dx^2}+k^2n^2(x)\psi(x)=\beta^2\psi(x), \tag{1}$$

where β is the propagation constant, k is the free space wave number and $n(x)$ is the refractive index profile

$$n(x)=\begin{cases} n_1, & \text{for } -a\le x\le a,\\ n_2, & \text{for } -b\le x<-a \text{ and } a<x\le b,\\ n_3, & \text{for } -\infty<x<-b \text{ and } b<x<\infty,\end{cases} \tag{2}$$

n_1, n_2 and n_3 are the refractive indices of the core, the cladding and the surrounding material, respectively ($n_1 > n_2 > n_3$), $2a$ and $2b$ are the thickness of the core and the cladding, respectively.

The exact normalized propagation constants for the even and odd modes can be found by applying the boundary conditions of continuity to the electric field (the even and odd modes were separated due to the symmetry of the waveguide) [2]

$$\tan(b_2a)-\frac{a_1a_2+a_2^2\tanh[a_2(b-a)]}{b_2a_2+b_2a_1\tanh[a_2(b-a)]}=0,\quad \text{for even modes}, \tag{3}$$

$$\cot(b_2a)+\frac{a_1a_2+a_2^2\tanh[a_2(b-a)]}{b_2a_2+b_2a_1\tanh[a_2(b-a)]}=0,\quad \text{for odd modes}, \tag{4}$$

where

$$a_1=\sqrt{\beta^2-(n_3k)^2},\ a_2=\sqrt{\beta^2-(n_2k)^2},$$
$$b_2=\sqrt{(n_1k)^2-\beta^2}. \tag{5}$$

The propagation constant $\beta_{p\infty}$ for the fundamental mode (m = 0) in the first-order perturbation of the infinitely clad waveguide [1]

$$\beta_{p\infty}^2=\beta_\infty^2-\left(\frac{V_2}{b}\right)^2\frac{\exp[-2W_1(\frac{b}{a}-1)]}{1+W_1\left(\frac{V_1}{U_1}\right)^2+\left(\frac{W_1}{U_1}\right)^2}$$
$$=\beta_\infty{}^2-\left(\frac{V_2U_1}{bV_1}\right)^2\frac{\exp[-2W_1(\frac{b}{a}-1)]}{1+W_1}, \tag{6}$$

is accurate if the wavelength is much shorter than the core of the waveguide (the intermediate expression is given to show an error in the work of Chiang [1] where there is a 0.5 instead of 1 in the denominator). In this relation β_∞ is given by the eigenvalue equation for the infinitely clad structure [1]

$$U_1\tan U_1=W_1, \tag{7}$$

where

$$U_1=a\sqrt{(n_1k)^2-\beta_\infty{}^2},W_1=a\sqrt{\beta_\infty{}^2-(n_2k)^2}, \tag{8}$$

$$V_1 = ak\sqrt{{n_1}^2 - {n_2}^2}, V_2 = bk\sqrt{{n_2}^2 - {n_3}^2}. \quad (9)$$

The propagation constant β_{pc} for the fundamental mode in the first-order perturbation of the coreless structure [1]

$$\beta_{pc}^2 = \beta_c^2 + \frac{V_1^2}{(1+\frac{W_2}{V_2^2}+\frac{U_2^2}{W_2V_2^2})a}[\frac{1}{b}+\frac{1}{aU_2}\sin(\frac{aU_2}{b})\cos(\frac{aU_2}{b})]$$

$$= \beta_c^2 + \frac{W_2V_1^2}{(1+W_2)a}[\frac{1}{b}+\frac{1}{aU_2}\sin(\frac{aU_2}{b})\cos(\frac{aU_2}{b})], \quad (10)$$

is accurate if the wavelength is much longer than the core of the waveguide (the intermediate expression is given to show an error in the work of Chiang [1] where there is a 2 in the last term from the denominator). In this relation β_c is given by the eigenvalue equation for the coreless structure [1]

$$U_2 \tan U_2 = W_2, \quad (11)$$

where

$$U_2 = b\sqrt{(n_2k)^2 - \beta_c^2}, W_2 = b\sqrt{\beta_c^2 - (n_3k)^2}, \; U_1^2 + W_1^2 = V_1^2, \; U_2^2 + W_2^2 = V_2^2. \quad (12)$$

The propagation constant β_{ts} for the fundamental mode in the two-sided perturbation method [1]

$$\beta_{ts}^2 = \beta_c^2 + \left(\frac{V_1}{a}\right)^2 \frac{F_1}{F_1+F_2+F_3}, \quad (13)$$

$$F_1 = \frac{T_1}{U_1^2 - \left(\frac{a}{b}\right)^2 U_2^2}, F_2 = \frac{T_1+T_2}{W_1^2 + \left(\frac{a}{b}\right)^2 U_2^2}, F_3 = \frac{T_2}{\left(\frac{a}{b}\right)^2 W_2^2 - W_1^2},$$

$$T_1 = W_1 \frac{\cos\left(\frac{aU_2}{b}\right)}{\cos(U_2)} - \frac{aU_2}{b}\frac{\sin\left(\frac{aU_2}{b}\right)}{\cos(U_2)}, T_2 = \left(\frac{a}{b}W_2 - W_1\right)\exp\left[-\left(\frac{b}{a}-1\right)W_1\right], \quad (14)$$

is accurate for large, small and medium a/λ.

The solution for the five-layer slab waveguide can thus be obtained from the solutions for three-layer slab structures [1].

Table 1. Comparison of normalized propagation constants P_m for no-core (nc), infinite cladding (∞c), two-sided perturbation (tsp), first-order perturbational (p_1), no-core first-order variational (v_1), no-core first-order variational (v_1^r) by taking U_{11} as the variational parameter and with a restriction (35), no-core first-order variational (v_1^{nr}) by taking U_{11} and W_{11} as the variational parameters and without a restriction (35), no-core second-order variational (v_2), no-core second-order variational (v_2^{nr}) by taking U_{11}, U_{22}, W_{11} and W_{22} as the variational parameters and without a restriction on the form (35), no-core third-order variational (v_3), no-core fourth-order variational (v_4), no-core fifth-order variational (v_5) approximations and exact value for all even modes TE_m at lasing wavelength of the Yb:YAG waveguide laser (a = 2μm, b = 4.5μm, n_1 = 1.8166, n_2 = 1.8154, n_3 = 1.755, λ = 1.029μm, p = 0.980190). The value of the parameter p is used as a criterion to separate the ranges of applicability for no-core and infinite cladding approximations.

	P_0	P_2	P_4	P_6	P_8
nc	0.967421	0.865573	0.663784	0.367992	0.012379
∞c	0.987341				
tsp	0.981048				
p_1,v_1	0.981646				
v_1^r	0.981705				
v_1^{nr}	0.981730				
v_2	0.982188	0.871749			
v_2^{nr}	0.982192	0.871795			
v_3	0.982189	0.871855	0.671916		
v_4	0.982201	0.871865	0.672079	0.376000	
v_5	0.982202	0.871865	0.672087	0.376058	0.017080
Exact	0.982205	0.871868	0.672095	0.376081	0.017303

In this paper we use the higher-order variational method to improve the accuracy of the propagation constants for the same sub–structures in an optical waveguide.

2. The variational method

The variational approximate solution (Eq. (1) can be written as an eigenvalue equation) of the scalar wave Eq. (1) is found by optimizing the functional (see, for example [3])

$$\begin{aligned} J = &\int_{-\infty}^{-b}[-f_1'^2 + (kn_3f_1)^2]dx + \int_{-b}^{-c}[-f_1'^2 + (kn_2f_1)^2]dx + \int_{-c}^{-a}[-f_2'^2 + (kn_2f_2)^2]dx \\ &+ \int_{-a}^{a}[-f_2'^2 + (kn_1f_2)^2]dx + \int_{a}^{c}[-f_2'^2 + (kn_2f_2)^2]dx + \int_{c}^{b}[-f_3'^2 + (kn_2f_3)^2]dx \\ &+ \int_{b}^{\infty}[-f_3'^2 + (kn_3f_3)^2]dx = 2\int_{b}^{\infty}[-f_3'^2 + (kn_3f_3)^2]dx + 2\int_{c}^{b}[-f_3'^2 + (kn_2f_3)^2]dx \\ &+ 2\int_{a}^{c}[-f_2'^2 + (kn_2f_2)^2]dx + \int_{-a}^{a}[-f_2'^2 + (kn_1f_2)^2]dx, \end{aligned} \quad (15)$$

with respect to the two parameters n_4 and c (n_4 is the refractive index of the optimized single-core structure and $2c$ is the thickness of this core), subject to the constraint that

$$I = \int_{-\infty}^{-c} f_1^2 dx + \int_{-c}^{c} f_2^2 dx + \int_{c}^{\infty} f_3^2 dx, \quad \beta^2 = \frac{J}{I}, \tag{16}$$

where the trial function is given by

$$f_1(x) = A\exp(W_4)\exp(W_4 \frac{x}{c}), \qquad x < -c,$$

$$f_2(x) = A\frac{\cos(U_4 \frac{x}{c})}{\cos(U_4)}, \text{ even modes}, \qquad -c \le x \le c,$$

$$f_2(x) = A\frac{\sin(U_4 \frac{x}{c})}{\sin(U_4)}, \text{ odd modes}, \qquad -c \le x \le c,$$

$$f_3(x) = A\exp(W_4)\exp(-W_4 \frac{x}{c}), \qquad x > c. \tag{17}$$

The eigenvalue equation (identical in form to Eq. (7)) for this optimized structure is given by

$$U_4 \tan U_4 = W_4, \tag{18}$$

where

$$U_4 = c\sqrt{(n_4 k)^2 - \beta_4^2}, W_4 = c\sqrt{\beta_4^2 - (n_3 k)^2}, V_4 = ck\sqrt{{n_4}^2 - {n_3}^2},$$

$$V_2 = bk\sqrt{{n_2}^2 - {n_3}^2}, \; V = ak\sqrt{{n_1}^2 - {n_2}^2}, \; U_4^2 + W_4^2 = V_4^2 \tag{19}$$

From Eqs. (15) - (19) we obtain the propagation constant β for the fundamental mode

$$\beta^2 = (n_2 k)^2 + \frac{W_4}{(1+W_4)c}\left(\frac{V^2}{a} - \frac{{U_4}^2}{c}\right) + \frac{W_4}{(1+W_4)U_4}\left(\frac{V}{a}\right)^2 \sin(\frac{aU_4}{c})\cos(\frac{aU_4}{c})$$
$$-\left(\frac{V_2 U_4}{bV_4}\right)^2 \frac{\exp[-2W_4(\frac{b}{c}-1)]}{1+W_4}. \tag{20}$$

In particular, for $c = a$, $n_3 = n_2$, $n_4 = n_1$, $\beta_4 = \beta_\infty$, $V = V_4 = V_1$, $W_4 = W_1$, $U_4 = U_1$, we obtain the propagation constant $\beta = \beta_{p\infty}$ (Eq. (6)) for the fundamental mode in the first-order perturbation of the infinitely clad waveguide. Also, for $c = b$, $n_4 = n_2$, $\beta_4 = \beta_c$, $U_4 = U_2$, $V_4 = V_2$, $W_4 = W_2$, $V = V_1$ we obtain the propagation constant $\beta = \beta_{pc}$ (Eq. (10)) for the fundamental mode in the first-order perturbation of the coreless structure.

The functional J_∞ and the propagation constant $\beta_{v\infty}$ for the fundamental mode in the variational method of the infinitely clad waveguide are given by

$$J_\infty = 2\int_b^\infty [-{f_3'}^2 + (kn_3 f_3)^2]dx + 2\int_a^b [-{f_3'}^2 + (kn_2 f_3)^2]dx + \int_{-a}^a [-{f_2'}^2 + (kn_1 f_2)^2]dx, \tag{21}$$

$$\beta_{v\infty}^2 = \frac{W_v}{1+W_v}[(kn_1)^2 + W_v\left(\frac{kn_1}{V_v}\right)^2 + \frac{1}{W_v}\left(\frac{kn_2 U_v}{V_v}\right)^2 - \left(\frac{U_v}{a}\right)^2]$$
$$+\frac{(n_3^2 - n_2^2)}{1+W_v}\left(\frac{kU_v}{V_v}\right)^2 \exp[-2W_v(\frac{b}{a}-1)], \tag{22}$$

where

$$U_v = a\sqrt{(kn_v)^2 - \beta_v^2}, W_v = a\sqrt{\beta_v^2 - (kn_2)^2}, \; V_v = ak\sqrt{{n_v}^2 - n_2^2}, \; U_v \tan(U_v) = W, \tag{23}$$

f_2 and f_3 are given in Eq. (15) with $c = a$, $U_4 = U_v$, $W_4 = W_v$, and n_v is a variational parameter. For $n_v = n_1$, $U_v = U_1$, $W_v = W_1$, $V_v = V_1$, we obtain $\beta_{v\infty} = \beta_{p\infty}$.

The functional J_c and the propagation constant β_{vc} for the fundamental mode in the variational method of the coreless structure are given by

$$J_c = 2\int_b^\infty [-{f_3'}^2 + (kn_3 f_3)^2]dx + 2\int_a^b [-{f_2'}^2 + (kn_2 f_2)^2]dx + \int_{-a}^a [-{f_2'}^2 + (kn_1 f_2)^2]dx, \tag{24}$$

$$\beta_{vc}^2 = \frac{W_v}{1+W_v}[(kn_2)^2 + W_v\left(\frac{kn_2}{V_v}\right)^2 + \frac{1}{W_v}\left(\frac{kn_3 U_v}{V_v}\right)^2 - \left(\frac{U_v}{b}\right)^2$$
$$+\frac{(n_1^2 - n_2^2)k^2 a}{b} + (n_1^2 - n_2^2)k^2 \frac{\sin\left(\frac{2aU_v}{b}\right)}{2U_v}], \tag{25}$$

where

$$U_v = b\sqrt{(kn_v)^2 - \beta_v^2}, W_v = b\sqrt{\beta_v^2 - (kn_3)^2}, V_v = bk\sqrt{{n_v}^2 - n_3^2}, \; U_v \tan(U_v) = W_v, \tag{26}$$

f_2 and f_3 are given in Eq. (15) with $c = b$, $U_4 = U_v$, $W_4 = W_v$, and n_v is a variational parameter. For $n_v = n_2$, $U_v = U_2$, $W_v = W_2$, $V_v = V_2$, we obtain $\beta_{vc} = \beta_{pc.}$

3. Higher-order variational method

In what follows we illustrate the application of the higher-order variational method to a five-layer slab waveguide that allow exact analytical solutions (the one-dimensional scalar-wave equation case).

The propagation constant β in the N-order variational method [4-5], satisfies the equation (from the relations (16) and (24) with $c = b$, $n_v = n_2$)

$$\det\begin{vmatrix} J_{11}-\beta^2 I_{11} & J_{12}-\beta^2 I_{12} & J_{13}-\beta^2 I_{13} & --- & J_{1N}-\beta^2 I_{1N} \\ J_{12}-\beta^2 I_{12} & J_{22}-\beta^2 I_{22} & J_{23}-\beta^2 I_{23} & --- & J_{2N}-\beta^2 I_{2N} \\ --- & --- & J_{33}-\beta^2 I_{33} & --- & --- \\ J_{1N\text{-}1}-\beta^2 I_{1N\text{-}1} & J_{2N\text{-}1}-\beta^2 I_{2N\text{-}1} & --- & --- & J_{N\text{-}1N}-\beta^2 I_{N\text{-}1N} \\ J_{1N}-\beta^2 I_{1N} & J_{2N}-\beta^2 I_{2N} & J_{3N}-\beta^2 I_{3N} & --- & J_{NN}-\beta^2 I_{NN} \end{vmatrix} = 0 \quad (27)$$

where I_{ii}, I_{ij}, J_{ii}, and J_{ij} for a coreless structure are given by (β_{ii} are the propagation constants in the no-core approximation)

$$I_{ij} = \int_{-\infty}^{-b} f_1(\beta_{ii}) f_1(\beta_{jj})dx + \int_{-b}^{b} f_2(\beta_{ii}) f_2(\beta_{jj})dx + \int_{b}^{\infty} f_3(\beta_{ii}) f_3(\beta_{jj})dx, \quad (28)$$

$$J_{ij} = 2\int_{b}^{\infty}\left[-f_3'(\beta_{ii})f_3'(\beta_{jj}) + (kn_3)^2 f_3(\beta_{ii})f_3(\beta_{jj})\right]dx$$
$$+2\int_{a}^{b}\left[-f_2'(\beta_{ii})f_2'(\beta_{jj}) + (kn_2)^2 f_2(\beta_{ii})f_2(\beta_{jj})\right]dx \quad (29)$$
$$+\int_{-a}^{a}\left[-f_2'(\beta_{ii})f_2'(\beta_{jj}) + (kn_1)^2 f_2(\beta_{ii})f_2(\beta_{jj})\right]dx,$$

$$I_{ii} = b\left[\frac{1}{W_{ii}} + \left(1+\frac{\sin(2U_{ii})}{2U_{ii}}\right)\sec^2(U_{ii})\right], \quad \text{even modes}, \quad (30)$$

$$I_{ii} = b\left[\frac{1}{W_{ii}} + \left(1-\frac{\sin(2U_{ii})}{2U_{ii}}\right)\csc^2(U_{ii})\right], \quad \text{odd modes}, \quad (30')$$

$$I_{ij} = b\left[\frac{2}{W_{ii}+W_{jj}} + \left(\frac{\sin(U_{ii}-U_{jj})}{U_{ii}-U_{jj}} + \frac{\sin(U_{ii}+U_{jj})}{U_{ii}+U_{jj}}\right)\sec(U_{ii})\sec(U_{jj})\right], \text{even modes}, \quad (31)$$

$$\mathrm{I}_{ij} = \mathrm{b}\left[\frac{2}{\mathrm{W}_{ii}+\mathrm{W}_{jj}} + \left(\frac{\sin(\mathrm{U}_{ii}-\mathrm{U}_{jj})}{\mathrm{U}_{ii}-\mathrm{U}_{jj}} - \frac{\sin(\mathrm{U}_{ii}+\mathrm{U}_{jj})}{\mathrm{U}_{ii}+\mathrm{U}_{jj}}\right)\csc(\mathrm{U}_{ii})\csc(\mathrm{U}_{jj})\right], \textit{odd modes}, \quad (31')$$

$$J_{ii} = \frac{\sec^2(U_{ii})}{2bU_{ii}W_{ii}}[(bkn_3)^2 U_{ii} - 2ab(kn_2)^2 U_{ii}W_{ii} + 2(bkn_2)^2 U_{ii}W_{ii} + 2ab(kn_1)^2 U_{ii}W_{ii}$$
$$-2U_{ii}^3 W_{ii} - U_{ii}W_{ii}^2 + U_{ii}\left((bkn_3)^2 - W_{ii}^2\right)\cos(2U_{ii}) + W_{ii}\left((bkn_2)^2 + U_{ii}^2\right)\sin(2U_{ii})$$
$$+W_{ii}(n_1^2 - n_2^2)(bk)^2 \sin(2aU_{ii}/b)], \text{ even modes}, \quad (32)$$

$$J_{ii} = \frac{\csc^2(U_{ii})}{2bU_{ii}W_{ii}}[(bkn_3)^2 U_{ii} - 2ab(kn_2)^2 U_{ii}W_{ii} + 2(bkn_2)^2 U_{ii}W_{ii} + 2ab(kn_1)^2 U_{ii}W_{ii}$$
$$-2U_{ii}^3 W_{ii} - U_{ii}W_{ii}^2 - U_{ii}\left((bkn_3)^2 - W_{ii}^2\right)\cos(2U_{ii}) - W_{ii}\left((bkn_2)^2 + U_{ii}^2\right)\sin(2U_{ii})$$
$$-W_{ii}(n_1^2 - n_2^2)(bk)^2 \sin(2aU_{ii}/b)], \text{ odd modes}, \quad (32')$$

$$J_{ij} = \frac{2[(bkn_3)^2 - W_{ii}W_{jj}]}{b(W_{ii}+Wjj)} - \frac{U_{ii}U_{jj}\sec(U_{ii})\sec(U_{jj})}{b}\left(\frac{\sin(a(U_{ii}-U_{jj})/b)}{U_{ii}-U_{jj}}\right.$$

$$-\frac{\sin(a(U_{ii}+U_{jj})/b)}{U_{ii}+U_{jj}}\Bigg)+b(kn_1)^2\sec(U_{ii})\sec(U_{jj})\Bigg(\frac{\sin(a(U_{ii}-U_{jj})/b)}{U_{ii}-U_{jj}}$$

$$+\frac{\sin(a(U_{ii}+U_{jj})/b)}{U_{ii}+U_{jj}}\Bigg)+\frac{2U_{ii}U_{jj}}{b(U_{ii}-U_{jj})(U_{ii}+U_{jj})}\{U_{ii}\tan(U_{jj})-U_{jj}\tan(U_{ii})$$

$$+\sec(U_{ii})\sec(U_{jj})\left[U_{jj}\sin(\frac{aU_{ii}}{b})\cos(\frac{aU_{jj}}{b})\text{-}U_{ii}\sin(\frac{aU_{jj}}{b})\cos(\frac{aU_{ii}}{b})\right]\}$$

$$+\frac{2b(kn_2)^2}{(U_{ii}-U_{jj})(U_{ii}+U_{jj})}\{U_{ii}\tan(U_{ii})\text{-}U_{jj}\tan(U_{jj})$$

$$+\sec(U_{ii})\sec(U_{jj})\left[U_{jj}\sin(\frac{aU_{jj}}{b})\cos(\frac{aU_{ii}}{b})\text{-}U_{ii}\sin(\frac{aU_{ii}}{b})\cos(\frac{aU_{jj}}{b})\right]\},\ \text{even modes}, \quad (33)$$

$$J_{ij}=\frac{2[(bkn_3)^2-W_{ii}W_{jj}]}{b(W_{ii}+Wjj)}-\frac{U_{ii}U_{jj}\csc(U_{ii})\csc(U_{jj})}{b}\Bigg(\frac{\sin(a(U_{ii}-U_{jj})/b)}{U_{ii}-U_{jj}}$$

$$+\frac{\sin(a(U_{ii}+U_{jj})/b)}{U_{ii}+U_{jj}}\Bigg)+b(kn_1)^2\csc(U_{ii})\csc(U_{jj})\Bigg(\frac{\sin(a(U_{ii}-U_{jj})/b)}{U_{ii}-U_{jj}}$$

$$-\frac{\sin(a(U_{ii}+U_{jj})/b)}{U_{ii}+U_{jj}}\Bigg)+\frac{2U_{ii}U_{jj}}{b(U_{ii}-U_{jj})(U_{ii}+U_{jj})}\{U_{jj}ctg(U_{ii})-U_{ii}ctg(U_{jj})$$

$$+\csc(U_{ii})\csc(U_{jj})\left[U_{ii}\sin(\frac{aU_{ii}}{b})\cos(\frac{aU_{jj}}{b})\text{-}U_{jj}\sin(\frac{aU_{jj}}{b})\cos(\frac{aU_{ii}}{b})\right]\}$$

$$+\frac{2b(kn_2)^2}{(U_{ii}-U_{jj})(U_{ii}+U_{jj})}\{U_{jj}ctg(U_{jj})\text{-}U_{ii}ctg(U_{ii})$$

$$+\csc(U_{ii})\csc(U_{jj})\left[U_{ii}\sin(\frac{aU_{jj}}{b})\cos(\frac{aU_{ii}}{b})\text{-}U_{jj}\sin(\frac{aU_{ii}}{b})\cos(\frac{aU_{jj}}{b})\right]\},\ \text{odd modes}, \quad (33')$$

$$U_{ii}=b\sqrt{(kn_2)^2-\beta_{ii}^{\ 2}},\ W_{ii}=b\sqrt{\beta_{ii}^{\ 2}\text{-}(kn_3)^2}, \quad (34)$$

$$U_{ii}\tan(U_{ii})=W_{\rm ii},U_{jj}\tan(U_{jj})=W_{\rm jj}. \quad (35)$$

The integrals in equations (28-29) with the chosen trial functions are evaluated analytically to reduce the amount of numerical computation.

Table 2. Comparison of normalized propagation constants P_m for no-core (nc), no-core first-order variational (v_1), no-core second-order variational (v_2), no-core third-order variational (v_3), no-core fourth-order variational (v_4) approximations and exact value for all odd modes TE_m at lasing wavelength of the Yb:YAG waveguide laser (a = 2 μm, b = 4.5 μm, n_1 = 1.8166, n_2 = 1.8154, n_3 = 1.755, λ = 1.029μm).

	P_1	P_3	P_5	P_7
nc	0.929162	0.776947	0.526936	0.190627
v_1	0.935655			
v_2	0.936027	0.786076		
v_3	0.936056	0.786203	0.533713	
v_4	0.936058	0.786205	0.533803	0.198512
Exact	0.936067	0.786208	0.533816	0.198630

For the comparison of effective refractive indices given by different methods, the normalized propagation constant $P_{\rm m}$ of the $TE_{\rm m}$-mode was found to be much more suitable than just the direct comparison of propagation constants [6]

$$P_{\rm m}=\frac{\left(\frac{\beta_m}{k}\right)^2-n_3^2}{n_1^2-n_3^2}\ ,\ 0<P_{\rm m}<1 \quad (36)$$

This normalized propagation constant is different from [1] where for $\beta/k = n_2$, $P=\frac{(\beta/k)^2-n_2^2}{n_1^2-n_2^2}=0$.

The criterion to separate the ranges of applicability for various approximations is determined by the value of the parameter p (in the place of $P = 0$ in [1])

$$p=\frac{n_2^2-n_3^2}{n_1^2-n_3^2}\ ,\ \begin{cases}P_{\rm m}>p, & \text{infinite-cladding approximation,}\\ P_{\rm m}<p, & \text{no-core approximation.}\end{cases} \quad (37)$$

To evaluate the exact normalized propagation constant for the five-layer slab symmetric waveguide laser we need to solve the eigenvalue equation (3-4).

4. Numerical results and conclusions

We have calculated the normalized propagation constants P_m for no-core (nc), infinite cladding (∞c), two-sided perturbation (tsp), first- order perturbational (p_1), no-core first-order variational (v_1), no-core second-order variational (v_2), no-core third-order variational (v_3), no-core fourth-order variational (v_4), no-core fifth-order variational (v_5) approximations and exact value for even (Tables 1, 3) and odd (Table 2) modes TE_m at lasing wavelength of the Yb:YAG waveguide laser [(a = 2μm, b = 4.5μm, n_1 = 1.8166, n_2 = 1.8154, n_3 = 1.755, λ = 1.029μm, 9 modes), (a = 4μm, b = 9μm, n_1 = 1.8166, n_2 = 1.8154, n_3 = 1.755, λ = 1.029μm, 17 modes)].

Table 3. Comparison of normalized propagation constants P_m for no-core (nc), infinite cladding (∞c), two-sided perturbation (tsp), first-order perturbational (p_1), no-core first-order variational (v_1), no-core first-order variational (v_1^r) by taking U_{11} as the variational parameter and with a restriction (35), no-core first-order variational (v_1^{nr}) by taking U_{11} and W_{11} as the variational parameters and without a restriction (35), no-core second-order variational (v_2), no-core second-order variational (v_2^{nr}) by taking U_{11}, U_{22}, W_{11} and W_{22} as the variational parameters and without a restriction on the form (35), no-core third-order variational (v_3), no-core fourth-ordervariational (v_4), no-core fifth-order variational (v_5) approximations and exact value for all even modes TE_m at lasing wavelength of the Yb:YAG waveguide laser (a = 4μm, b = 9μm, n_1 = 1.8166, n_2 = 1.8154, n_3 = 1.755, λ = 1.029μm, [2]).

	P_0	P_2	P_4	P_6	P_8
nc	0.976751 [6]	0.949254	0.894321	0.812087	0.702794
∞c	0.993199 [6]				
tsp	0.992674 [6]				
p_1,v_1 v_1^r v_1^{nr}	0.991367 0.991495 0.991515				
v_2 v_2^{nr}	0.992906 0.992920	0.954552 0.954697			
v_3	0.992912	0.955006	0.902844		
v_4	0.992945	0.955078	0.903421	0.819873	
v_5	0.992948	0.955073	0.903442	0.820122	0.710603
Exact	0.992955 [6]	0.955090	0.903464	0.820138	0.710951

Also, we have calculated the normalized propagation constants P_m for no-core (nc), infinite cladding (∞c), two-sided perturbation (tsp), first-order perturbational (p_1), no-core first-order variational (v_1), no-core second-order variational (v_2), no-core third-order variational (v_3), no-core fourth-order variational (v_4), no-core fifth-order variational (v_5) approximations and exact value for even (Table 4) and odd (Table 5) modes TE_m at lasing wavelength of the Nd:YAG waveguide laser (a = 10μm, b = 15μm, n_1 = 1.8151, n_2 = 1.8147, n_3 = 1.755, λ = 1.064μm, 27 modes).

The coreless approximation gives the same number of modes as the exact method. The value of the parameter p is used as a criterion to separate the ranges of applicability for no-core and infinite cladding approximations (relation 37). In the higher-order variational method, for a large number of the modes, the optimum value of the normalized propagation constant corresponds to the minimum difference between values of β from two successive orders (the exact value of the physical quantity β does not depend of the order of the variational method). The accuracy of the method depends on how many eigenfunctions (17) are used. The normalized propagation constants for even modes were separated from the odd modes due to the symmetry of the waveguide.

Table 4. Comparison of normalized propagation constants P_m for no-core (nc), infinite cladding (∞c), two-sided perturbation (tsp), first-order perturbational (p_1), no-core first-order variational (v_1), no-core first-order variational (v_1^r) by taking U_{11} as the variational parameter and with a restriction (35), no-core first-order variational (v_1^{nr}) by taking U_{11} and W_{11} as the variational parameters and without a restriction (35), no-core second-order variational (v_2), no-core second-order variational (v_2^{nr}) by taking U_{11}, U_{22}, W_{11} and W_{22} as the variational parameters and without a restriction on the form (35), no-core third-order variational (v_3), no-core fourth-order variational (v_4) approximations and exact value for all even modes TE_m at lasing wavelength of the Nd:YAG waveguide laser (a = 10μm, b = 15μm, n_1 = 1.8151, n_2 = 1.8147, n_3 = 1.755, λ = 1.064μm [2], p = 0.993233).

	P_0	P_2	P_4	P_6
nc	0.991837 [6]	0.980666	0.958331	0.924843
∞c	0.998460 [6]			
tsp	0.998160 [6]			
p_1,v_1 v_1^r v_1^{nr}	0.998157 0.988166 0.998169			
v_2 v_2^{nr}	0.998239 0.998252	0.984879 0.984929		
v_3	0.998256	0.985153	0.962133	
v_4	0.998259	0.985168	0.962250	0.929412
Exact	0.998259 [6]	0.985179	0.962254	0.929491

Table 5. Comparison of normalized propagation constants P_m for no-core (nc), no-core first-order variational (v_1), no-core second-order variational (v_2), no-core third-order variational (v_3), no-core fourth-order variational (v_4) approximations and exact value for all odd modes TE_m at lasing wavelength of the Nd:YAG waveguide laser (a = 10μm, b = 15 μm, n_1 = 1.8151, n_2 = 1.8147, n_3 = 1.755, λ = 1.064μm).

	P_1	P_3	P_5	P_7
nc	0.987647	0.970893	0.942980	0.903924
v_1	0.992925			
v_2	0.993148	0.974562		
v_3	0.993176	0.974766	0.947255	
v_4	0.993177	0.974772	0.947328	0.908404
Exact	0.993181	0.974779	0.947332	0.908499

Table 6 . The calculated variational refractive index n_v^o and the normalized propagation constant P_0 for the fundamental mode TE_0 (single mode) in multi-quantum-well waveguide as a function of odd number of quantum wells n (for n = 1 we are near cut-off).

n	n_v^o	P_0	P_0 (Exact)
1	3.306339	0.001212	0.001212
3	3.312920	0.006909	0.006909
5	3.314841	0.016964	0.016964
51	3.318046	0.414741	0.414741
53	3.318058	0.424321	0.424321
55	3.318069	0.433350	0.433350

The applicability of the formulas depends on the near and far cut-off regimes and also on the ratio a/λ and the refractive index differences. The calculation accuracy for the high-order modes is not high in the case $n_v = n_2$ (see the last column of Table 1). Our results can be further improved by using an optimized variational parameter $n_v = n_v^o$ for $c = b$. As an example we choose a multiple quantum well optical waveguide [6] at wavelength λ = 1.55 μm with 55 quantum wells, each of thickness L_w = 7 nm, interspersed with barriers (56 barriers) of width L_b = 12 nm. The refractive indices of the quantum wells (GaAs), barriers ($Al_{0.18}Ga_{0.82}As$) and cladding materials ($Al_{0.32}Ga_{0.68}As$) are n_1 = 3.3704, n_2 = 3.2874 and n_3 = 3.2224, respectively. Fig. 1 and Table 6 show the calculated variational refractive index n_v^o and the normalized propagation constant P_0 for the fundamental mode TE_0 (single mode) in multi-quantum-well waveguide as a function of odd number of quantum wells.

Also, we can improve the results by taking U_{ii}, U_{jj}, W_{ii} and W_{jj} as the variational parameters (which depend on the refractive index and the thickness of the core for an optimized single-core structure) for no-core higher-order variational (v_i^r, v_i^{nr}, i = 1,2,…) with (v_i^r) and without (v_i^{nr}) a restriction (35), related to the gradient continuity of the approximate trial solution (Tables 1, 3, 4 for v_1^r, v_1^{nr} and v_2^{nr}). The exact solution for *TE* mode requires as the electric field E_y and $\partial E_y / \partial x$ to be continuous at all boundaries (across the planes – *b*, - *a*, *a* and *b*). In our no-core approximation these conditions for the trial function are limited to the planes – *b* and *b*. Thus we can to explain the better results in the absence of the relation (35) but with a new variational parameter.

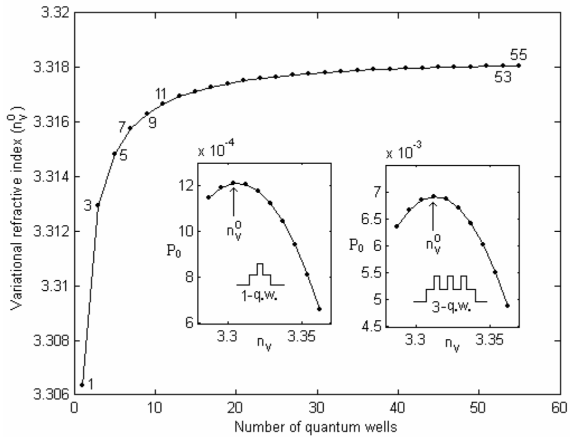

Fig. 1. The variational refractive index n_v^o for the fundamental mode TE_0 (single mode) in multi-quantum-well waveguide as a function of odd number of quantum wells n. The inset is the normalized propagation constant P_0 as a function of refractive index n_v for a quantum well (1-q.w.) and for three quantum wells (3–q.w.).

Our values of the normalized propagation constant P_m (by using higher-order variational method), are closer to the exact value in comparison with the first-order perturbation of the infinite clad waveguide, first–order perturbation of the coreless structure and the two–sided perturbation methods[1].

The higher-order variational method can be applied in analyzing the propagation constants of the supermodes in a coupled waveguide system and for a complicated multiple quantum well waveguide.

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