JOURNAL OF OPTOELECTRONICS AND ADVANCED MATERIALS Vol. 18, Iss. 3-4, March – April 2016, p. 383-388

Lie symmetry analysis and exact solutions of the time fractional gas dynamics equation

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Finding the symmetries of a given fractional differential equation is a hot topic in the field of fractional differentiation and its applications. In this manuscript, the Lie symmetries of the time fractional gas dynamics (TFGD) equation are analyzed and new exact solutions are obtained.

(Received July 13, 2015; accepted April 05, 2016)

Keywords: Fractional gas dynamics equation, Lie symmetry method, Erdélyi-Kober fractional differential operator

1. Introduction

The fractional calculus represents a generalization of the classical one and it started to become very popular in several branches of science and engineering. We recall that in acoustics, electro-chemistry, electromagnetics, control processing, anomalous diffusion and visco-elasticity some phenomena are better described by using the fractional differential equations [1, 2, 3, 4, 5, 6, 7, 8].

As it is known the conservation laws play a very important role in physics and engineering from both theoretical and practical viewpoints. We recall that the laws of conservation of energy, angular momentum and linear momentum play key roles in solving many problems appearing in mathematical physics. Special analytical solutions for both ordinary differential equations (ODEs) and partial differential equations (PDEs) can be extracted using a systematic process, namely, Lie groups [9, 10, 11, 12, 13]. This method requires the calculation of variable transformations which leave a differential equation form invariant. Lie symmetries were introduced in order to solve ordinary differential equations. We recall that by using the symmetry method we can reduce the systems of differential equations and we can find the equivalent systems of differential equations of simpler form (reduction process). Also symmetry groups can be used for classifying different symmetry classes of solutions. According to Nöether's theorem, every continuous symmetry of a physical system corresponds to a conservation law of the system. Even though Lie symmetry method has been extensively applied to find the exact solutions of a range of classical PDEs and ODEs, it was applied for few fractional differential equations. Thus, an important task in the fractional calculus area is to find the Lie symmetries and the exact solutions for the fractional differential equations. We recall that the fractional derivatives are nonlocal operators, therefore there exists a huge motivation to find the symmetries of some equations, e.g. the time fractional gas dynamics, corresponding to the real world phenomena. Moreover, the fractional order models, in some cases, they gave better results than the integer order models, therefore this is another motivation to find the symmetries of the fractional gas dynamics equation. Motivated by the importance of the studied equation and taking into account that the fractional generalization were generalized recently only for the time fractional derivative, we consider the following equation [14, 15]:

$$\partial_t^\alpha u + \frac{1}{2}\left(u^2\right)_x - u\left(1-u\right) = 0,\ 0 < \alpha \le 1, \qquad (1)$$

where $\partial_t^\alpha u := D_t^\alpha u$ stands for Riemann-Liouville derivative of order α which the range of applicability of the Riemann-Liouville is more general than Caputo derivative. We recall that the Riemann-Liouville derivative is defined by [1, 2, 3, 4]

$$D_t^\alpha u(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\dfrac{\partial^n}{\partial t^n}\displaystyle\int_0^t (t-\xi)^{n-\alpha-1} u(x,\xi)d\xi, & n = [\alpha]+1 \\ \dfrac{\partial^n u}{\partial t^n}, & \alpha = n \in \mathrm{N} \end{cases} \qquad (2)$$

where $\Gamma(z)$ is the Euler Gamma function and $u = u(x,t)$ is a function of the spatial coordinate x and time t. When $\alpha = 1$, the TFGD equation reduces to the classical gas dynamics equation which is considered as a case study for solving hyperbolic conservation laws because it depicts the next level of complexity after the Burger's equation.

The organization of the manuscript is given below:

In Section 2 we present the point symmetries of the time fractional partial differential equations of first order. The description of the Lie symmetry analysis of the equation (1) is shown in Section 3. Also, the general similarity forms and the symmetry reductions are established. In Section 4 the exact solutions of the TFGD equation are investigated. The conclusion part ends our manuscript.

2. Point symmetries of the fractional partial differential equations

We consider the fractional partial differential equations (FPDE) as [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]:

$$\Delta(x,t,u,u_x,\partial_t^\alpha u)=0,\ \ 0<\alpha<1. \tag{3}$$

The infinitesimal generator V for Eq. (3) is written in the following form:

$$V=\xi^t(x,t,u)\frac{\partial}{\partial t}+\xi^x(x,t,u)\frac{\partial}{\partial x}+\eta(x,t,u)\frac{\partial}{\partial u}, \tag{4}$$

which satisfies the symmetry condition:

$$Pr^{(\alpha,1)}V(\Delta)\,|_{\Delta=0}=0. \tag{5}$$

The prolongation operator $Pr^{(\alpha,1)}V$ takes the form

$$Pr^{(\alpha,1)}V=V+\eta^x\partial_{u_x}+\eta_\alpha^0\partial_{\partial_t^\alpha u}, \tag{6}$$

where

$$\begin{aligned}\eta^x&=D_x(\eta)-u_xD_x(\xi^x)-u_tD_x(\xi^t),\\ \eta_\alpha^0&=D_t^\alpha(\eta)+\xi^xD_t^\alpha(u_x)-D_t^\alpha(\xi^xu_x)\\ &+D_t^\alpha(D_t(\xi^t)u)-D_t^{\alpha+1}(\xi^tu)+\xi^tD_t^{\alpha+1}(u),\end{aligned}$$

and the operator D_t^α denotes the total fractional derivative operator. By using the invariance condition and the conservative property of Riemann-Liouville fractional operator we have

$$\xi^t(x,t,u)\,|_{t=0}=0. \tag{7}$$

Making use of the fractional Leibniz rule, we can rewrite η_α^0 as following

$$\eta_\alpha^0=D_t^\alpha(\eta)-\alpha D_t(\xi^t)\frac{\partial^\alpha u}{\partial t^\alpha}-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi^x)D_t^{\alpha-n}(u_x)-\sum_{n=1}^{\infty}\binom{\alpha}{n+1}D_t^{n+1}(\xi^t)D_t^{\alpha-n}(u). \tag{8}$$

From the chain rule [30]

$$\frac{d^mf(g(t))}{dt^m}=\sum_{k=0}^{m}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}[-g(t)]^r\frac{d^m}{dt^m}[g^{k-r}(t)]\times\frac{d^kf(g)}{dg^k} \tag{9}$$

and setting $f(t)=1,$ we get

$$D_t^\alpha(\eta)=\frac{\partial^\alpha\eta}{\partial t^\alpha}+\eta_u\frac{\partial^\alpha u}{\partial t^\alpha}-u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}+\sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n}D_t^{\alpha-n}(u)+\vartheta, \tag{10}$$

where

$$\vartheta=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}[u^{k-r}]\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}. \tag{11}$$

Hence

$$\eta_\alpha^0=\frac{\partial^\alpha\eta}{\partial t^\alpha}+\left(\eta_u-\alpha D_t(\xi^t)\right)\frac{\partial^\alpha u}{\partial t^\alpha}-u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}+\vartheta+\sum_{n=1}^{\infty}[\binom{\alpha}{n}\frac{\partial^\alpha\eta_u}{\partial t^\alpha}-\binom{\alpha}{n+1}D_t^{n+1}(\xi^t)]D_t^{\alpha-n}(u)-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi^x)D_t^{\alpha-n}(u_x).$$

3. Lie theory for TFGD equation

According to the Lie symmetries and utilizing the prolongation $Pr^{(\alpha,1)}V$ to the Eq. (1), it is possible to get the following invariance criterion:

$$\eta_\alpha^0+\eta u_x+u\eta^x-\eta+2\eta u=0. \tag{12}$$

Using (8) and (12), one can obtain the determining equations for the symmetry group of Eq. (1) and solution of these equations concludes the symmetries:

$$\xi^t=c_2t,\quad \xi^x=c_1+2\alpha c_2x,\quad \eta=\alpha c_2u, \tag{13}$$

given in the vector forms:

$$V_1=\frac{\partial}{\partial x},\ \ V_2=t\frac{\partial}{\partial t}+2\alpha x\frac{\partial}{\partial x}+\alpha u\frac{\partial}{\partial u}. \tag{14}$$

For the symmetry of V_2, the corresponding similarity variables are:

$$\zeta = xt^{-2\alpha}, \; u(x,t) = t^{\alpha}\mathrm{F}(\zeta). \qquad (15)$$

Now, by using the transformations (15), we reduce the TFGD (1) into a fractional ordinary differential equation (FODE) as it can be seen from the following theorem:

Theorem 3.1: The transformation (15) reduces TFGD equation to the following nonlinear fractional ordinary differential equation:

$$\left(\mathrm{P}_{\frac{1}{2\alpha}}^{1,\alpha}\mathrm{F}\right)(\zeta) - \mathrm{F}(\mathrm{F}'+\mathrm{F}-1) = 0, \qquad (16)$$

with the Erdélyi-Kober fractional differential operator $\mathrm{P}_{\beta}^{\tau,\alpha}$ of form:

$$\left(\mathrm{P}_{\beta}^{\tau,\alpha}\mathrm{F}\right) := \prod_{j=0}^{n-1}\left(\tau + j - \frac{1}{\beta}\zeta\frac{d}{d\zeta}\right)\left(\mathrm{K}_{\beta}^{\tau+\alpha,n-\alpha}\mathrm{F}\right)(\zeta),$$

$$n = \begin{cases}[\alpha]+1, & \alpha \notin \mathrm{N}\\ \alpha, & \alpha \in \mathrm{N}\end{cases} \qquad (17)$$

where

$$\left(\mathrm{K}_{\beta}^{\tau,\alpha}\mathrm{F}\right)(\zeta) := \begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^{\infty}(u-1)^{\alpha-1}u^{-(\tau+\alpha)}\mathrm{F}(\zeta u^{\frac{1}{\beta}})du, & \alpha > 0,\\ \mathrm{F}(\zeta), & \alpha = 0,\end{cases} \qquad (18)$$

is the Erdélyi-Kober fractional integral operator.

Proof: From definition of the Riemann-Liouville fractional derivative we have

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\sigma)^{n-\alpha-1}\sigma^{\alpha}\mathrm{F}\left(x\sigma^{-2\alpha}\right)d\sigma\right],$$

$$n-1 < \alpha < n, \; n = 1,2,... \qquad (19)$$

Setting $v = \frac{t}{\sigma}$, one can get $d\sigma = -\frac{t}{v^2}dv$, therefore (19) can be written as

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^n}{\partial t^n}\left[t^n\left(\mathrm{K}_{\frac{1}{2\alpha}}^{1+\alpha,n-\alpha}\mathrm{F}\right)(\zeta)\right]. \qquad (20)$$

Taking into account the relation $(\zeta = xt^{-2\alpha})$, we conclude

$$t\frac{\partial}{\partial t}\phi(\zeta) = t\frac{\partial\zeta}{\partial t}\frac{d\phi(\zeta)}{d\zeta} = -2\alpha\zeta\frac{d\phi(\zeta)}{d\zeta}. \qquad (21)$$

As a result, we have

$$\frac{\partial^n}{\partial t^n}\left[t^n\left(\mathrm{K}_{\frac{1}{2\alpha}}^{1+\alpha,n-\alpha}\mathrm{F}\right)(\zeta)\right] = \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^n\left(\mathrm{K}_{\frac{1}{2\alpha}}^{1+\alpha,n-\alpha}\mathrm{F}\right)(\zeta)\right)\right]$$

$$= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-1}\left(n - 2\alpha\zeta\frac{d}{d\zeta}\right)\left(\mathrm{K}_{\frac{1}{2\alpha}}^{1+\alpha,n-\alpha}\mathrm{F}\right)(\zeta)\right]$$

$$= ...$$

$$= \prod_{j=0}^{n-1}\left(1 + j - 2\alpha\zeta\frac{d}{d\zeta}\right)\left(\mathrm{K}_{\frac{1}{2\alpha}}^{1+\alpha,n-\alpha}\mathrm{F}\right)(\zeta) = \left(\mathrm{P}_{\frac{1}{2\alpha}}^{1,\alpha}\mathrm{F}\right)(\zeta), \qquad (22)$$

which completes the proof.

4. Some exact solutions of TFGD equation

In this section, we introduce a transformation to reduce the TFGD equation into a nonlinear ODE. Then using the reduced equation we extract some exact solutions of TFGD equation of traveling wave types and transformation

$$u(x,t) = \Psi(\xi), \; \xi = \frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)} + \mu x, \qquad (23)$$

where λ and μ are constants, allows us to reduce the Eq. (1) into a first order ODE as follows:

$$\Psi'(\xi) = \frac{\Psi(\xi)(1-\Psi(\xi))}{\lambda + \mu\Psi(\xi)}. \qquad (24)$$

This equation is nonlinear and it is not possible to find a general solution for arbitrary parameters λ and μ. Therefore, to find the exact solutions of Eq. (24), we consider some special cases:

- $\lambda = \mu$:

In this case an explicit solution of Eq. (24) can be derive as:

$$\Psi(\xi) = 1 + \frac{1}{2}\left(1 \pm \sqrt{1 + 4e^{\frac{\xi + c_1}{\mu}}}\right)e^{-\left(\frac{\xi + c_1}{\mu}\right)}, \qquad (25)$$

or equivalently from (23) we conclude:

$$u(x,t) = 1 + \frac{1}{2}\left(1 \pm \sqrt{1 + 4e^{\left(x + \frac{c_1}{\mu} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)}}\right)e^{-\left(x + \frac{c_1}{\mu} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)}, \qquad (26)$$

where c_1 is integration constant. Now, we plot the probability density function $u(x,t)$ for different time fractional Brownian motions $\alpha = 0.25, 0.5, 0.75, 1,$ for various space values in Fig. 1 and various times in Fig. 2. The parameters have been selected as $c_1 = \mu = 1$. It is obvious from Fig. 1 that $u(x,t)$ increases for $t \in [0,1.2)$ and when α increase. Reverse behaviour occurs for u and other larger times. From Fig. 2. the solution increases for different time values when fractional order decrease.

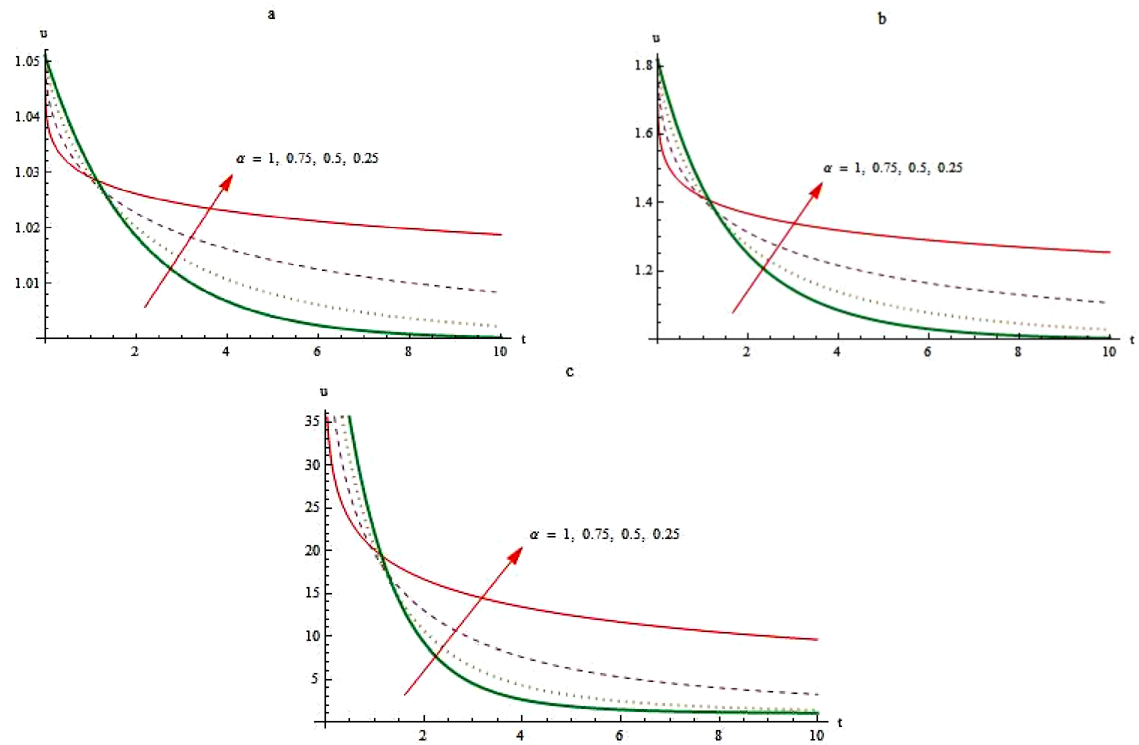

Fig. 1. Profile of solution (26) with different values of fractional differential order at (a) x=5, (b) x=0 and (c) x=-5

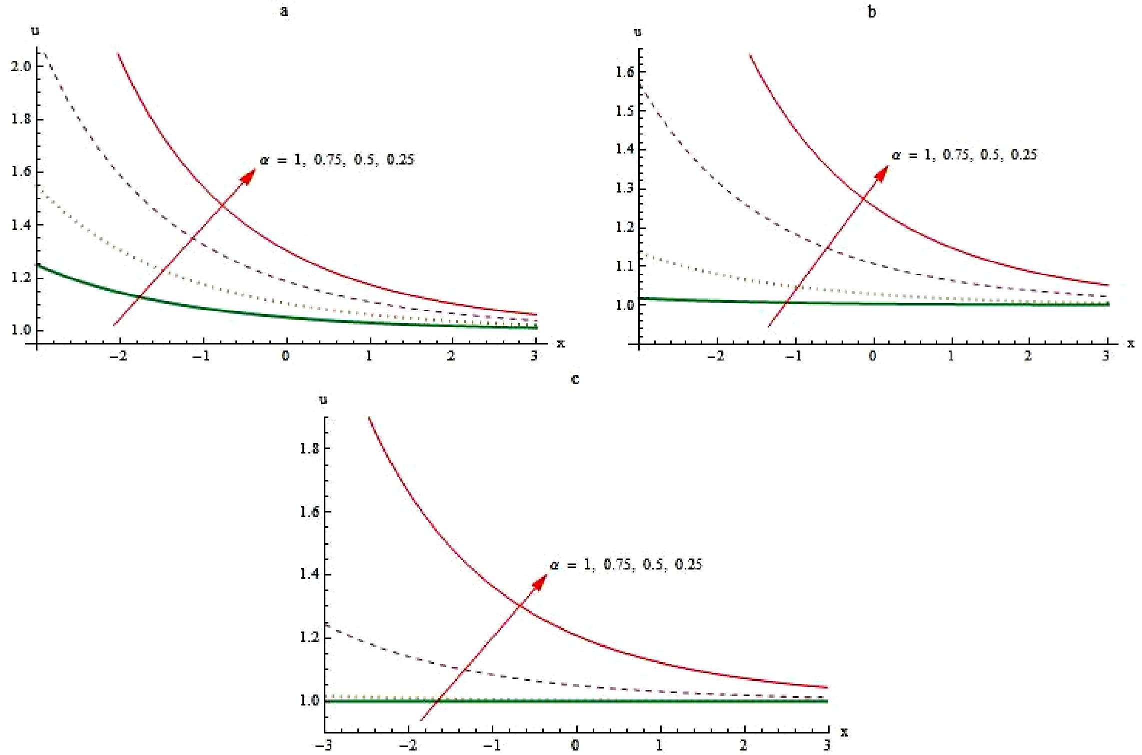

Fig. 2. Profile of solution (26) with different values of fractional differential order at (a) t=5, (b) t=10 and (c) t=20.

- $\lambda = -\mu$:

In this case, another explicit solution is as following:

$$\Psi(\xi) = c_1 e^{-\frac{\xi}{\mu}}, \qquad (27)$$

and therefore

$$u(x,t) = c_1 e^{\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - x\right)}. \qquad (28)$$

- $\lambda = -\frac{\mu}{2}$:

In this case the exact solution of Eq. (24) is expressed as

$$\Psi(\xi) = \frac{1 \pm \sqrt{1 + 4e^{-2\left(\frac{\xi + c_1}{\mu}\right)}}}{2}, \tag{29}$$

or equivalently

$$u(x,t) = \frac{1 \pm \sqrt{1 + 4e^{\frac{-2}{\mu}\left(\mu x + c_1 - \frac{\mu t^{\alpha}}{2\Gamma(1+\alpha)}\right)}}}{2}. \tag{30}$$

- $\lambda = -2\mu$:

Here, a solution of the form:

$$\Psi(\xi) = \frac{e^{-\frac{\xi + c_1}{\mu}} \pm \sqrt{e^{-2\left(\frac{\xi + c_1}{\mu}\right)} - 4e^{-\left(\frac{\xi + c_1}{\mu}\right)}}}{2}, \tag{31}$$

can be derived and (23) gives another solution of (1). Unlike the previous obtained solution, in Fig. 3. when α increases the solution decreases for $t \in [0,1.2)$ and reverse behaviour occur for larger time values. Finally, according to Fig. 4. the solution increases when fractional order increases.

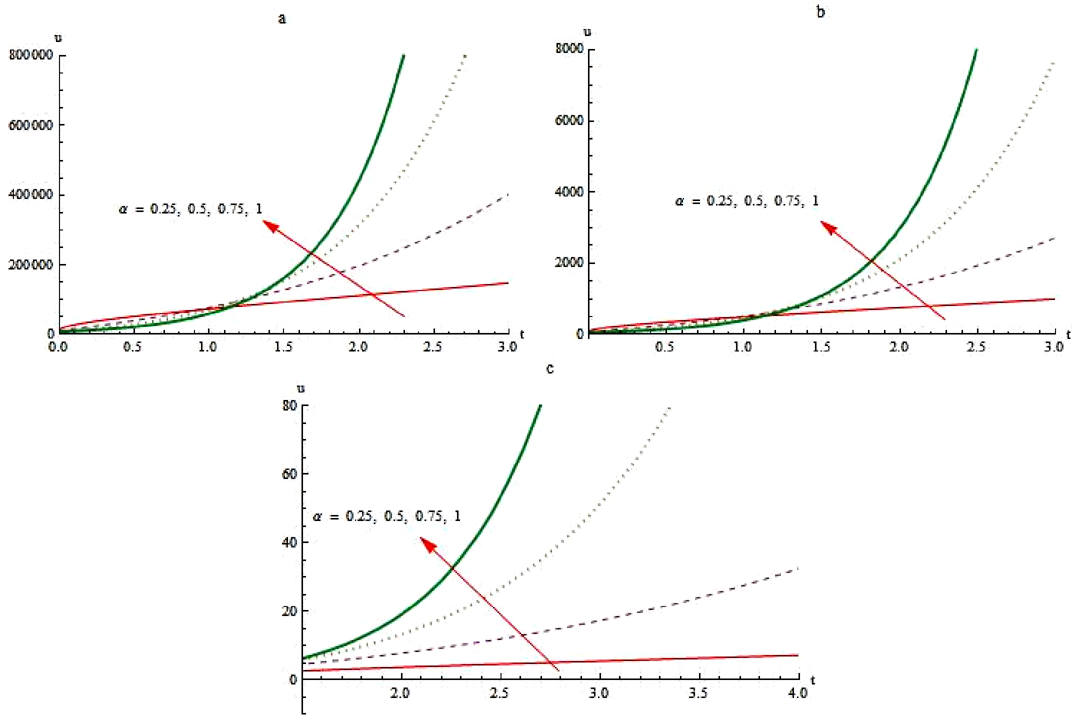

Fig. 3. Profile of solution (31) with different values of fractional differential order at (a) x=-10, (b) x=-5 and (c) x=0

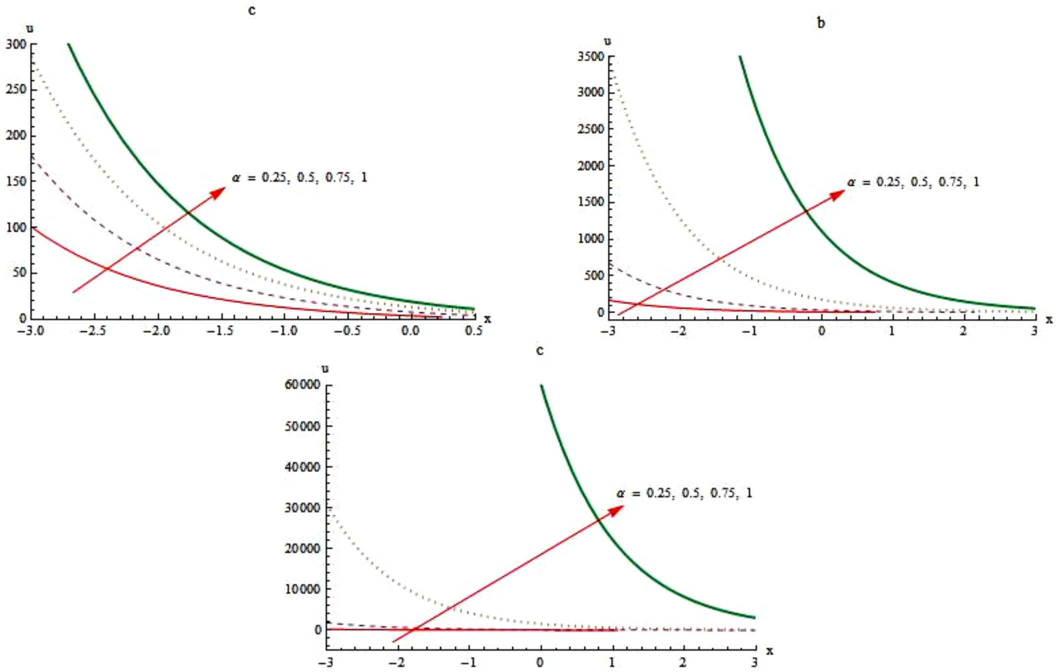

Fig. 4. Profile of solution (31) with different values of fractional differential order at (a) t=2, (b) t=4 and (c) t=6.

5. Conclusion

The method of Lie symmetries is successfully applied to investigate the symmetry properties and similarity reductions of time fractional gas dynamics equation. We have demonstrated that Eq. (1) is reducible into a first order nonlinear ODE of fractional order with Erdélyi-Kober kind. Exact solutions of traveling wave types are extracted.

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