Optical solitons in (2+1)-dimensions with dual-power law nonlinearity

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This paper proposes four kinds new optical soliton solutions in (2+1)-dimensions with dual-power law nonlinearity. These are bright, dark, singular and of mixed type of solitons. The conditions on the physical parameters for the existence of the obtained structures are also presented. These solutions are helpful to recognize physical phenomena described by the governing equation.

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1. Introduction

It is well known that the propagation of optical pulses in a monomode optical fiber is described by the nonlinear Schrödinger (NLS) equation with cubic and higher nonlinearity. Studying the properties of the NLS family of equations has grown steadily in recent years because of their potential applications in many fields of physics, including nonlinear optics, water waves, Bose-Einstein condensates, biomolecular dynamics, and nonlinear quantum field theory [1-20]. For picosecond light pulses, this model includes only the group velocity dispersion (GVD) and the self-phase modulation, and it admits bright and dark soliton-type pulse propagation in anomalous and normal dispersion regimes, respectively [8]. The unique property of optical solitons, either bright or dark, is their particle-like behavior in interaction [9].

Early works mostly concentrated on Kerr-type media, where the refractive index varies linearly with the pulse intensity I as $n = n_0 + n_2 I$, with n_0 and n_2 being the linear refractive index coefficient and the cubic nonlinearity coefficient which is related to the third-order susceptibility $\chi^{(3)}$ as $n_2 = 3\chi^{(3)}/8n_0$, respectively. To enlarge the information carrying capacity, it is necessary

to transmit ultrashort optical pulse of subpicosecond and femtosecond size [10]. To produce ultrashort pulses, the intensity of the incident light field increases, which leads to non-Kerr nonlinearities, changing the physical feature of the system [11].

It is interesting that the propagation of ultrashort pulses in dual-power law media can be described by the following NLS equation in (1+2) dimensions [12]-[15]:

$$iq_t + \frac{1}{2}\left(q_{xx} + q_{yy}\right) + \left(|q|^{2m} + k|q|^{4m}\right)q = 0, \tag{1}$$

where $q(x, y, t)$ is the complex envelope of the electric field, and k is a constant.

In this equation, the first term represents the evolution term, the second and third terms, in parenthesis, represent the dispersion in x and y directions while the fourth and fifth terms in parenthesis together represents nonlinearity.

Knowledge of the exact traveling wave solutions to the NLS equation and its extensions is important from many points of view (e.g., for the calculation of certain important physical quantities analytically as well as serving as diagnostics for simulations). In the following we propose specific ansatz solutions for obtaining novel exact solutions of Eq. (1). We particularly find new types of exact soliton solutions of the bright, dark and singular type under certain parametric conditions.

2. Mathematical analysis

We start the analysis by assuming a solution given by the following phase amplitude format [15, 16]

$$q(x,y,t) = P(x,y,t)e^{i\phi}, \tag{2}$$

where P is the amplitude portion while ϕ is the phase portion of the soliton. It is also assumed that [15, 17]

$$\phi(x,t) = -\kappa_1 x - \kappa_2 x + \omega t + \theta, \tag{3}$$

where κ_1 and κ_2 represents the soliton frequency in the x and y directions respectively, while ω represents the solitary wave number and finally θ is the phase constant of the soliton.

Substituting (2) in (1) and separating out the real and imaginary parts, yields the following pair of relations:

$$\frac{\partial P}{\partial t} - \kappa_1 \frac{\partial P}{\partial x} - \kappa_2 \frac{\partial P}{\partial y} = 0, \tag{4}$$

and

$$\left\{\omega + \frac{1}{2}\left(\kappa_1^2 + \kappa_2^2\right)\right\} P - \frac{1}{2}\frac{\partial^2 P}{\partial x^2} - \frac{1}{2}\frac{\partial^2 P}{\partial y^2} - P^{2m+1} - kP^{4m+1} = 0. \tag{5}$$

For obtaining closed form solutions for these equations, we propose new ansatz solutions that are different from the one used in Ref. [12] as follows:

(i) A $\cosh^2$ ansatz:

$$P = \frac{A}{\left(D + \cosh^2 \tau\right)^p}, \tag{6}$$

(ii) A $\sinh^2$ ansatz:

$$P = \frac{A}{\left(D + \sinh^2 \tau\right)^p}, \tag{7}$$

(iii) A combined $\cosh - \sinh$ ansatz:

$$P = \frac{A}{\left(D + \rho \cosh \tau + \lambda \sinh \tau\right)^p}, \tag{8}$$

(iv) A tanh ansatz:

$$P = \left(\sigma + \eta \tanh \tau\right)^p, \tag{9}$$

where

$$\tau = B_1 x + B_2 y - vt. \tag{10}$$

Here in (6)-(9), A is the amplitude of the soliton, B_1 is the inverse width in the x-direction and B_2 is the inverse width in the y-direction and v represents the velocity of the soliton. Also the exponent p will be determined in terms of m and the constants $A, D, \rho, \lambda, \sigma$ and η will also be determined in terms of k, B_1 and B_2.

By employing these ansatz solutions we will obtain new structures illustrating the potentially rich set of soliton solutions for the nonlinear Schrödinger equation with dual-power law nonlinearity. To our knowledge, the ansatze (6)-(9) have not been previously used to obtain exact analytic soliton solutions of the NLS equation in (2 +1) dimensions with dual-power law nonlinearity equation (1).

2.1. Soliton solutions

In this section, we will use the ansatze presented above to develop soliton solutions to the NLS equation with dual-power law nonlinearity (1).

2.1.1. Ansatz-I

Let us first apply the rational ansatz I for solving Eqs. (4) and (5). Using (6) and (10), Eq. (4) reduces to

$$\frac{Apv \cosh \tau \sinh \tau}{\left(D + \cosh^2 \tau\right)^{p+1}} + \frac{Ap\left(\kappa_1 B_1 + \kappa_2 B_2\right)\cosh \tau \sinh \tau}{\left(D + \cosh^2 \tau\right)^{p+1}} = 0, \tag{11}$$

while (5) reduces to

$$\begin{aligned}
&-\left\{\omega + \frac{1}{2}\left(\kappa_1^2 + \kappa_2^2\right)\right\}\frac{A}{\left(D + \cosh^2 \tau\right)^p} \\
&+ \frac{2p^2 AB_1^2}{\left(D + \cosh^2 \tau\right)^p} - \frac{p(2p+1)(2D+1)AB_1^2}{\left(D + \cosh^2 \tau\right)^{p+1}} \\
&+ \frac{2Dp(p+1)(D+1)AB_1^2}{\left(D + \cosh^2 \tau\right)^{p+2}} + \frac{2p^2 AB_2^2}{\left(D + \cosh^2 \tau\right)^p} \\
&- \frac{p(2p+1)(2D+1)AB_2^2}{\left(D + \cosh^2 \tau\right)^{p+1}} + \frac{2Dp(p+1)(D+1)AB_2^2}{\left(D + \cosh^2 \tau\right)^{p+2}} \\
&+ \frac{A^{2m+1}}{\left(D + \cosh^2 \tau\right)^{p(2m+1)}} + \frac{kA^{4m+1}}{\left(D + \cosh^2 \tau\right)^{p(4m+1)}} = 0.
\end{aligned} \tag{12}$$

From (11), it can be seen that

$$v = -\kappa_1 B_1 - \kappa_2 B_2, \tag{13}$$

Notice that in order to compensate for the dispersive effects with nonlinearity, the following analytical condition must hold

$$p = \frac{1}{2m}, \tag{14}$$

which can be obtained by equating the exponents $(4m+1)p$ and $p+2$ in (12). This same value of p is recovered, when the exponents $(2m+1)p$ and $p+1$ are set equal to one another. Also noting that the functions $1/\left(D + \cosh^2 \tau\right)^{p+j}$ for $j = 0,1$ and 2 are linearly independent, their respective coefficients in (12) must vanish. Therefore, these yield the following relations:

$$\omega=\frac{1}{2m^2}\left[B_1^2+B_2^2-m^2\left(\kappa_1^2+\kappa_2^2\right)\right], \tag{15}$$

$$A=\left[\frac{(m+1)(2D+1)\left(B_1^2+B_2^2\right)}{2m^2}\right]^{1/2m}, \tag{16}$$

and

$$D=\frac{1}{2}\left[\sqrt{\frac{m^2(2m+1)}{m^2(2m+1)+2k(m+1)^2\left(B_1^2+B_2^2\right)}}-1\right], \tag{17}$$

Now from (17), the domain restriction of D implies

$$k\in\left(-\frac{m^2(2m+1)}{2(m+1)^2\left(B_1^2+B_2^2\right)},\ \infty\right), \tag{18}$$

which shows that solitons for the NLSE in 1+2 dimensions in a parabolic law regime exists for k to lie in the neighborhood given in (18). Thus, finally, the new bright soliton solution of (1) is given by

$$q(x,y,t)=\frac{A}{\left[D+\cosh^2\left(B_1x+B_2y-vt\right)\right]^{1/2m}}\times e^{i(-\kappa_1x-\kappa_2x+\omega t+\theta)}, \tag{19}$$

where the amplitude A is related to the widths B_1 and B_2 as given by (16), the wave number ω is given by (15) and the velocity is given by (13).

2.1.2. Ansatz-II

Let us now consider the rational ansatz II for solving Eqs. (4) and (5). Using (7) and (10), Eq. (4) reduces to

$$\frac{2Apv\cosh\tau\sinh\tau}{\left(D+\sinh^2\tau\right)^{p+1}}+\frac{2Ap\left(\kappa_1B_1+\kappa_2B_2\right)\cosh\tau\sinh\tau}{\left(D+\sinhh^2\tau\right)^{p+1}}=0, \tag{20}$$

while (5) reduces to

$$\begin{aligned}&-\left\{\omega+\frac{1}{2}\left(\kappa_1^2+\kappa_2^2\right)\right\}\frac{A}{\left(D+\sinh^2\tau\right)^p}\\&+\frac{2p^2AB_1^2}{\left(D+\sinh^2\tau\right)^p}-\frac{p(2p+1)(2D-1)AB_1^2}{\left(D+\sinhh^2\tau\right)^{p+1}}\\&+\frac{2Dp(p+1)(D-1)AB_1^2}{\left(D+\sinh^2\tau\right)^{p+2}}+\frac{2p^2AB_2^2}{\left(D+\sinh^2\tau\right)^p}\\&-\frac{p(2p+1)(2D-1)AB_2^2}{\left(D+\sinh^2\tau\right)^{p+1}}+\frac{2Dp(p+1)(D-1)AB_2^2}{\left(D+\sinh^2\tau\right)^{p+2}}\\&+\frac{A^{2m+1}}{\left(D+\sinh^2\tau\right)^{p(2m+1)}}+\frac{kA^{4m+1}}{\left(D+\sinh^2\tau\right)^{p(4m+1)}}=0.\end{aligned} \tag{21}$$

By virtue of balancing principle, on equating the exponents $(4m+1)p$ and $p+2$, from (21), gives

$$p=\frac{1}{2m}. \tag{22}$$

Now (20) gives

$$v=-\kappa_1B_1-\kappa_2B_2. \tag{23}$$

Then, from (21) by equating the coefficients of the linearly independent functions $1/\left(D+\sinh^2\tau\right)^{p+j}$ for $j=0,1$ and 2 to zero yields the relations

$$\omega=\frac{1}{2m^2}\left[B_1^2+B_2^2-m^2\left(\kappa_1^2+\kappa_2^2\right)\right], \tag{24}$$

$$A=\left[\frac{(m+1)(2D-1)\left(B_1^2+B_2^2\right)}{2m^2}\right]^{1/2m}, \tag{25}$$

and

$$D=\frac{1}{2}\left[\sqrt{\frac{m^2(2m+1)}{m^2(2m+1)+2k(m+1)^2\left(B_1^2+B_2^2\right)}}+1\right], \tag{26}$$

Now from (26), the domain restriction of D implies

$$k\in\left(-\frac{m^2(2m+1)}{2(m+1)^2\left(B_1^2+B_2^2\right)},\ \infty\right). \tag{27}$$

Hence, finally, the new singular-type soliton solution to (1) is given by

$$q(x,y,t)=\frac{A}{\left[D+\sinh^2\left(B_1x+B_2y-vt\right)\right]^{1/2m}}\times e^{i(-\kappa_1x-\kappa_2x+\omega t+\theta)}, \tag{28}$$

where the amplitude A is related to the widths B_1 and B_2 as given by (25), the wave number ω is given by (24) and the velocity is given by (23).

2.1.3. Ansatz-III

Now we take the rational ansatz III for solving Eqs. (4) and (5). Using (8) and (10), Eq. (4) reduces to

$$\frac{Apv(\rho\sinh\tau+\lambda\cosh\tau)}{\left(D+\rho\cosh\tau+\lambda\sinh\tau\right)^{p+1}}+\frac{Ap\left(\kappa_1B_1+\kappa_2B_2\right)(\rho\sinh\tau+\lambda\cosh\tau)}{\left(D+\rho\cosh\tau+\lambda\sinh\tau\right)^{p+1}}=0, \tag{29}$$

while (5) reduces to

$$
\begin{aligned}
&-\left\{\omega+\frac{1}{2}\left(\kappa_1^2+\kappa_2^2\right)\right\}\frac{A}{\left(D+\rho\cosh\tau+\lambda\sinh\tau\right)^p}\\
&+\frac{AB_1^2p^2}{2\left(D+\rho\cosh\tau+\lambda\sinh\tau\right)^p}-\frac{ADB_1^2p\left(2p+1\right)}{2\left(D+\rho\cosh\tau+\lambda\sinh\tau\right)^{p+1}}\\
&+\frac{AB_1^2p\left(p+1\right)\left(D^2-\rho^2+\lambda^2\right)}{2\left(D+\rho\cosh\tau+\lambda\sinh\tau\right)^{p+2}}+\frac{AB_2^2p^2}{2\left(D+\rho\cosh\tau+\lambda\sinh\tau\right)^p}\\
&-\frac{ADB_2^2p\left(2p+1\right)}{2\left(D+\rho\cosh\tau+\lambda\sinh\tau\right)^{p+1}}+\frac{AB_2^2p\left(p+1\right)\left(D^2-\rho^2+\lambda^2\right)}{2\left(D+\rho\cosh\tau+\lambda\sinh\tau\right)^{p+2}}\\
&+\frac{A^{2m+1}}{\left(D+\rho\cosh\tau+\lambda\sinh\tau\right)^{p(2m+1)}}+\frac{kA^{4m+1}}{\left(D+\rho\cosh\tau+\lambda\sinh\tau\right)^{p(4m+1)}}=0.
\end{aligned} \tag{30}
$$

By balancing principle, we obtain

$$p=\frac{1}{2m}. \tag{31}$$

Now (29) gives

$$v=-\kappa_1B_1-\kappa_2B_2. \tag{32}$$

Now from (30), setting the coefficients of the linearly independent functions $1/\left(D+\rho\cosh\tau+\lambda\sinh\tau\right)^{p+j}$ to zero, where $j=0,1,2,$ gives

$$\omega=\frac{1}{8m^2}\left[B_1^2+B_2^2-4m^2\left(\kappa_1^2+\kappa_2^2\right)\right], \tag{33}$$

$$A=\left[\frac{D\left(m+1\right)\left(B_1^2+B_2^2\right)}{4m^2}\right]^{1/2m}, \tag{34}$$

and

$$D=\left[\frac{2m^2\left(2m+1\right)\left(\rho^2-\lambda^2\right)}{2m^2\left(2m+1\right)+k\left(m+1\right)^2\left(B_1^2+B_2^2\right)}\right]^{1/2}, \tag{35}$$

Now from (35), the domain restriction of D implies

$$k\in\left(-\frac{2m^2\left(2m+1\right)}{\left(m+1\right)^2\left(B_1^2+B_2^2\right)},\ \infty\right). \tag{36}$$

Hence, finally, the new bright-type soliton solution to (1) is given by

$$
\begin{aligned}
q\left(x,y,t\right)=&\frac{A}{\left[D+\rho\cosh\left(B_1x+B_2y-vt\right)+\lambda\sinh\left(B_1x+B_2y-vt\right)\right]^{1/2m}}\\
&\times e^{i\left(-\kappa_1x-\kappa_2x+\omega t+\theta\right)},
\end{aligned} \tag{37}
$$

where the amplitude A is related to the widths B_1 and B_2 as given by (34), the wave number ω is given by (33) and the velocity is given by (32).

2.1.4. Ansatz-IV

Now we take the ansatz IV for solving Eqs. (4) and (5). Using (9) and (10), Eq. (4) reduces to

$$
\begin{aligned}
&\frac{pv}{\eta}\left[\left(\eta^2-\sigma^2\right)\left(\sigma+\eta\tanh\tau\right)^{p-1}+2\sigma\left(\sigma+\eta\tanh\tau\right)^p-\left(\sigma+\eta\tanh\tau\right)^{p+1}\right]\\
&+\frac{Ap\left(\kappa_1B_1+\kappa_2B_2\right)}{\eta}\begin{bmatrix}\left(\eta^2-\sigma^2\right)\left(\sigma+\eta\tanh\tau\right)^{p-1}+2\sigma\left(\sigma+\eta\tanh\tau\right)^p\\-\left(\sigma+\eta\tanh\tau\right)^{p+1}\end{bmatrix}=0,
\end{aligned} \tag{38}
$$

while (5) reduces to

$$
\begin{aligned}
&-\left\{\omega+\frac{1}{2}\left(\kappa_1^2+\kappa_2^2\right)\right\}\left(\sigma+\eta\tanh\tau\right)^p\\
&+\frac{pB_1^2}{2\eta^2}\left\{\begin{aligned}&\left(p-1\right)\left(\eta^2-\sigma^2\right)^2\left(\sigma+\eta\tanh\tau\right)^{p-2}\\&+2\sigma\left(2p-1\right)\left(\eta^2-\sigma^2\right)\left(\sigma+\eta\tanh\tau\right)^{p-1}\\&+2p\left(3\sigma^2-\eta^2\right)\left(\sigma+\eta\tanh\tau\right)^p\\&-2\sigma\left(2p+1\right)\left(\sigma+\eta\tanh\tau\right)^{p+1}\\&-\left(p+1\right)\left(\sigma+\eta\tanh\tau\right)^{p+2}\end{aligned}\right\}\\
&+\frac{pB_2^2}{2\eta^2}\left\{\begin{aligned}&\left(p-1\right)\left(\eta^2-\sigma^2\right)^2\left(\sigma+\eta\tanh\tau\right)^{p-2}\\&+2\sigma\left(2p-1\right)\left(\eta^2-\sigma^2\right)\left(\sigma+\eta\tanh\tau\right)^{p-1}\\&+2p\left(3\sigma^2-\eta^2\right)\left(\sigma+\eta\tanh\tau\right)^p\\&-2\sigma\left(2p+1\right)\left(\sigma+\eta\tanh\tau\right)^{p+1}\\&-\left(p+1\right)\left(\sigma+\eta\tanh\tau\right)^{p+2}\end{aligned}\right\}\\
&+\left(\sigma+\eta\tanh\tau\right)^{p(2m+1)}+k\left(\sigma+\eta\tanh\tau\right)^{p(4m+1)}=0.
\end{aligned} \tag{39}
$$

Proceeding as before, we obtain

$$p=\frac{1}{2m}, \tag{40}$$

$$v=-\kappa_1B_1-\kappa_2B_2, \tag{41}$$

$$\omega=\frac{1}{2m^2}\left[B_1^2+B_2^2-m^2\left(\kappa_1^2+\kappa_2^2\right)\right], \tag{42}$$

$$\sigma=\pm\eta=\frac{\left(m+1\right)\left(B_1^2+B_2^2\right)}{2m^2}, \tag{43}$$

and

$$k=\frac{m^2\left(2m+1\right)}{2\left(m+1\right)^2\left(B_1^2+B_2^2\right)}. \tag{44}$$

For this case, the novel dark solution to (1) is given by

$$q\left(x,y,t\right)=\sigma^{1/2m}\left[1\pm\tanh\left(B_1x+B_2y-vt\right)\right]^{1/2m}e^{i\left(-\kappa_1x-\kappa_2x+\omega t+\theta\right)}, \tag{45}$$

where the free parameters σ and η are given by (43) while the velocity and the wave number are given by (41) and (42), respectively.

3. Conclusion

In conclusion, we have obtained new analytical soliton solutions for the nonlinear Schrödinger equation with dual-power law nonlinearity in (2+1) dimensions by means of specific ansatze. Bright, dark and singular envelope solutions of the model have been derived by adopting four types of ansatz solutions. The soliton parameters are determined in term of the physical parameters involved in the governing equation. Conditions for the existence of propagating envelopes have also been reported. These solutions are helpful for understanding physical phenomena arising in dual-power law media.

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