Optical solitons in cascaded system with spatio-temporal dispersion by ansatz approach

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In this paper, the cascaded system is revisited with Kerr and power laws of nonlinearity. The spatio-temporal dispersion is included this time in order to make the model of study well-posed. Bright, dark and singular soliton solutions are obtained along with respective constraints. These integrability conditions must hold for the solitons to exist in a cascaded system.

(Received November 22, 2014; accepted January 21, 2015)

Keywords: Solitons, Cascaded system, Integrability

1. Introduction

Optical solitons is one of the most fascinating areas of research in the field of nonlinear optics. The study of bright, dark and singular solitons is of equal interest in this field. In the past few decades, there are a plethora of research results that are reported in several books and journals [1-20]. The main focus in those reports is polarization preserving fibers. It is equally important to focus on vector models that are studied in the context of birefringent fibers, Thirring solitons, cascaded systems, DWDM systems, optical couplers and many other areas.

This paper will focus on obtaining soliton solutions to cascaded system where the model is considered with spatio-temporal dispersion (STD) in addition to group-velocity dispersion (GVD). The inclusion of STD makes the model well-posed as indicated during 2012 [9, 12]. This model was studied earlier where cascaded system was integrated without STD. This paper is therefore a generalized version of earlier reported results. The governing equation is coupled nonlinear Schrödinger's equation (NLSE). The integrability aspect of the model will be focus of the paper. Bright, dark and singular soliton solutions to the model will be obtained.

2. The model

The dynamics of solitons in cascaded system is governed by coupled NLSE. The corresponding system in dimensionless form of this coupled NLSE is given by

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + c_1 F(|r|^2)q = 0 \quad (1)$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + c_2 F(|q|^2)r + d_2 F(|r|^2)r = 0 \quad (2)$$

Equations (1) and (2) represents the coupled NLSE where the functional F represents nonlinear media. This paper will only address Kerr law nonlinearity, also known as cubic nonlinearity and power law nonlinear medium. The variables $q(x,t)$ and $r(x,t)$ are the two components of the wave profile in a cascaded system. The first terms in both components are the linear evolution terms. The coefficients of a_j for $j=1,2$ represents GVD while the coefficients of b_j for $j=1,2$ are the STD terms. Finally, the coefficients of c_j are the nonlinear terms that could be either Kerr or power law nonlinearity.

This paper will focus the integrability aspect of the above model given by (1) and (2). While several algorithms of integrability exist in the literature, this paper will focus on the ansatz approach. This will lead to exact 1-soliton solution to the cascaded system with STD. There will be a few constraint conditions that will appear. These constraints guarantee the integrability of the model. Bright, dark and singular soliton solutions will be retrieved for this model. The general integration scheme will be discussed in the next section. This will be followed by further details for the two forms of nonlinearity.

3. Integration scheme

To integrate the coupled NLSE (1)-(2) the starting assumption is a solution of the form

$$q(x,t) = P_1(x,t)e^{i\phi_1(x,t)} \tag{3}$$

$$r(x,t) = P_2(x,t)e^{i\phi_2(x,t)} \tag{4}$$

where $P_l(x,t)$, for $l = 1, 2$, represents the amplitude component of the soliton solution, while the phase factor is given by

$$\phi_j(x,t) = -\kappa_j x + \omega_j t + \theta_j \tag{5}$$

where $j = 1, 2$. Here κ_j is the frequency of the solitons while ω_j represents the wave number and θ_j is the phase constant. Substituting (3)–(5) into (1) and (2) and then decomposing into real and imaginary parts gives

$$\begin{aligned} &a_1 \frac{\partial^2 P_1}{\partial x^2} + b_1 \frac{\partial^2 P_1}{\partial x \partial t} \\ &+ (b_1\omega_1\kappa_1 - \omega_1 - a_1\kappa_1^2)P_1 + c_1 F(P_2^2)P_1 = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} &a_2 \frac{\partial^2 P_2}{\partial x^2} + b_2 \frac{\partial^2 P_2}{\partial x \partial t} + (b_2\omega_2\kappa_2 - \omega_2 - a_2\kappa_2^2)P_2 \\ &+ c_2 F(P_1^2)P_2 + d_2 F(P_2^2)P_2 = 0 \end{aligned} \tag{7}$$

for the real portion, and for the imaginary part equations one have

$$(1 - \kappa_1 b_1)\frac{\partial P_1}{\partial t} - (2a_1\kappa_1 - b_1\omega_1)\frac{\partial P_1}{\partial x} = 0 \tag{8}$$

$$(1 - \kappa_2 b_2)\frac{\partial P_2}{\partial t} - (2a_2\kappa_2 - b_2\omega_2)\frac{\partial P_2}{\partial x} = 0 \tag{9}$$

The imaginary parts (8) and (9) lead to the speed of the solitons as

$$v = -\frac{2a_1\kappa_1 - b_1\omega_1}{1 - b_1\kappa_1} \tag{10}$$

$$v = -\frac{2a_2\kappa_2 - b_2\omega_2}{1 - b_2\kappa_2} \tag{11}$$

as long as the constraints

$$b_1\kappa_1 \neq 1 \tag{12}$$

$$b_2\kappa_2 \neq 1 \tag{13}$$

are satisfied. It must be noted that $P(x,t)$ can be represented as $g(x - vt)$ where the function g is the soliton wave profile depending on the type of nonlinearity, and v is the speed of the soliton. Now, equating the two expressions for the soliton speed (10) and (11) leads to a constraint relation between the soliton parameters as

$$\begin{aligned} &(1 - b_2\kappa_2)(2a_1\kappa_1 - b_1\omega_1) \\ &= (1 - b_1\kappa_1)(2a_2\kappa_2 - b_2\omega_2) \end{aligned} \tag{14}$$

This constraint relation holds for both Kerr and power laws of nonlinearity as well as for bright, dark and singular solitons for all of of these two laws of nonlinearity. The real part equations given by (6) and (7) will now be analyzed separately in the next two sub-sections for the mentioned nonlinearities.

3.1 Kerr law

Kerr nonlinearity originates when light wave in an optical fiber is subjected to nonlinear response. In this case, $F(s) = s$ and consequently the model is (1)–(2) change to

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + c_1 |r|^2 q = 0 \tag{15}$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + c_2 |q|^2 r + d_2 |r|^2 r = 0 \tag{16}$$

The imaginary parts (8) and (9) are preserved as well as relations (10)–(14). For this case, the corresponding real parts (6) and (7) become

$$\begin{aligned} &a_1 \frac{\partial^2 P_1}{\partial x^2} + b_1 \frac{\partial^2 P_1}{\partial x \partial t} \\ &+ (b_1\omega_1\kappa_1 - \omega_1 - a_1\kappa_1^2)P_1 + c_1 P_2^2 P_1 = 0 \end{aligned} \tag{17}$$

$$\begin{aligned} &a_2 \frac{\partial^2 P_2}{\partial x^2} + b_2 \frac{\partial^2 P_2}{\partial x \partial t} \\ &+ (b_2\omega_2\kappa_2 - \omega_2 - a_2\kappa_2^2)P_2 \\ &+ c_2 P_1^2 P_2 + d_2 P_2^3 = 0 \end{aligned} \tag{18}$$

The ansatz approach will be applied to this pair of equations to retrieve the corresponding bright, dark and singular soliton solutions.

3.1.1 Bright solitons

In this section, the case where both components (17) and (18) support bright solitons is considered. Thus, the assumption for the wave profile is

$$P_1 = A_1 \operatorname{sech}^{p_1}\tau \tag{19}$$

and

$$P_2 = A_2 \operatorname{sech}^{p_2}\tau \tag{20}$$

with

$$\tau = B(x - vt) \tag{21}$$

where A_j for $j = 1, 2$ and B represent respectively the amplitude and inverse width of the soliton. As was stated previously, v stands for the soliton speed. Substitution of (19) and (20) reduce (17) and (18) to

$$\begin{aligned}&\left\{\begin{matrix}(b_1\omega_1\kappa_1 - \omega_1 - a_1\kappa_1^2)\\ + p_1^2(a_1 - b_1 v)B^2\end{matrix}\right\}\operatorname{sech}^{p_1}\tau\\ &- p_1(1+p_1)(a_1 - b_1 v)B^2\operatorname{sech}^{p_1+2}\tau\\ &+ c_1A_2^2\operatorname{sech}^{p_1+2p_2}\tau = 0\end{aligned} \tag{22}$$

and

$$\begin{aligned}&\left\{\begin{matrix}(b_2\omega_2\kappa_2 - \omega_2 - a_2\kappa_2^2)\\ + p_2^2(a_2 - b_2 v)B^2\end{matrix}\right\}\operatorname{sech}^{p_2}\tau\\ &- p_2(1+p_2)(a_2 - b_2 v)B^2\operatorname{sech}^{p_2+2}\tau\\ &+ c_2A_1^2\operatorname{sech}^{p_2+2p_1}\tau + d_2A_2^2\operatorname{sech}^{3p_2}\tau = 0\end{aligned} \tag{23}$$

Next, from (22), equating the exponent pair $(p_1 + 2p_2,\ p_1 + 2)$ while from (23), equating the exponent pairs $(2p_1 + p_2,\ p_2 + 2)$ and $(3p_2,\ p_2 + 2)$ leads to

$$p_1 = p_2 = 1 \tag{24}$$

Setting the coefficients of the linearly independent functions, $\operatorname{sech}^{p_j+l}\tau$ where $j = 1, 2$ and $l = 0, 2$, to zero in (22) and (23) leads to the wave numbers of the solitons being given by

$$\omega_1 = \frac{c_1A_2^2 - 2a_1\kappa_1^2}{2(1 - b_1\kappa_1)} \tag{25}$$

and

$$\omega_2 = \frac{(c_2A_1^2 + d_2A_2^2) - 2a_2\kappa_2^2}{2(1 - b_2\kappa_2)} \tag{26}$$

subject to the constraints (12) and (13). Similarly, with the aid of (10) and (11) the width of the solitons can be retrieve as

$$B = A_2\left[\frac{c_1(1 - b_1\kappa_1)}{2(a_1 + a_1b_1\kappa_1 - b_1^2\omega_1)}\right]^{\frac{1}{2}} \tag{27}$$

and

$$B = \left[\frac{(1 - b_2\kappa_2)(c_2A_1^2 + d_2A_2^2)}{2(a_2 + a_2b_2\kappa_2 - b_2^2\omega_2)}\right]^{\frac{1}{2}} \tag{28}$$

which in turn introduce the constraints

$$c_1(1 - b_1\kappa_1) \times (a_1 + a_1b_1\kappa_1 - b_1^2\omega_1) > 0 \tag{29}$$

$$\begin{aligned}&(1 - b_2\kappa_2)(c_2A_1^2 + d_2A_2^2)\\ &\times (a_2 + a_2b_2\kappa_2 - b_2^2\omega_2) > 0\end{aligned} \tag{30}$$

By equating the two values of the width B from (27) and (28) one have

$$\frac{A_1}{A_2} = \left[\frac{c_1(1 - b_1\kappa_1)(a_2 + a_2b_2\kappa_2 - b_2^2\omega_2)}{c_2(1 - b_2\kappa_2)(a_1 + a_1b_1\kappa_1 - b_1^2\omega_1)} - \frac{d_2}{c_2}\right]^{\frac{1}{2}} \tag{31}$$

which is valid whenever

$$\begin{aligned}&\begin{pmatrix}c_1(1 - b_1\kappa_1)(a_2 + a_2b_2\kappa_2 - b_2^2\omega_2)\\ - d_2(1 - b_2\kappa_2)(a_1 + a_1b_1\kappa_1 - b_1^2\omega_1)\end{pmatrix}\\ &\times c_2(1 - b_2\kappa_2)(a_1 + a_1b_1\kappa_1 - b_1^2\omega_1) > 0\end{aligned} \tag{32}$$

Thus the bright 1-soliton solution to the cascaded system with STD (15)–(16) is given by

$$q(x,t) = A_1 \operatorname{sech}[B(x - vt)]e^{i(-\kappa_1 x + \omega_1 t + \theta_1)} \tag{33}$$

and

$$r(x,t) = A_2 \operatorname{sech}[B(x - vt)]e^{i(-\kappa_2 x + \omega_2 t + \theta_2)} \tag{34}$$

The constraint conditions that must remain valid in order for the bright soliton to exist are (12), (13), (29), (30) and (32).

3.1.2 Dark solitons

For dark solitons the assumption for the wave profile of the two components is

$$P_1 = A_1 \tanh^{p_1}\tau \tag{35}$$

and

$$P_2 = A_2 \tanh^{p_2}\tau \tag{36}$$

where τ is being defined as in (21) . However for dark solitons the parameters A_1 , A_2 and B are considered herein as free parameters and v correspond to the dark soliton speed. Substitution of (35) and (36) into (17) and (18) lead to

$$\begin{aligned}&\left\{\begin{matrix}(b_1\omega_1\kappa_1-\omega_1-a_1\kappa_1^2)\\-2p_1^2(a_1-b_1v)B^2\end{matrix}\right\}\tanh^{p_1}\tau\\&+p_1(p_1+1)(a_1-b_1v)B^2\tanh^{p_1+2}\tau\\&+p_1(p_1-1)(a_1-b_1v)B^2\tanh^{p_1-2}\tau\\&+c_1A_2^2\tanh^{p_1+2p_2}\tau=0\end{aligned}\tag{37}$$

and

$$\begin{aligned}&\left\{\begin{matrix}(b_2\omega_2\kappa_2-\omega_2-a_2\kappa_2^2)\\-2p_2^2(a_2-b_2v)B^2\end{matrix}\right\}\tanh^{p_2}\tau\\&+p_2(p_2+1)(a_2-b_2v)B^2\tanh^{p_2+2}\tau\\&+p_2(p_2-1)(a_2-b_2v)B^2\tanh^{p_2-2}\tau\\&+c_2A_1^2\tanh^{p_2+2p_1}\tau+d_2A_2^2\tanh^{3p_2}\tau=0\end{aligned}\tag{38}$$

From (37), equating the exponent pair $(p_1+2p_2,\ p_1+2)$ and from (38), equating the exponents $(3p_2,\ p_2+2)$ leads to (24). The stand alone linearly independent functions are $\tanh^{p_j-2}\tau$ for $j=1,2$ whose coefficients, when set to zero, also lead to (24). Thus, from (37) and (38), setting the coefficients of the linearly independent functions $\tanh^{p_j+l}\tau$ for $j=1,2$ and $l=0,2$, to zero gives the wave number of the dark solitons as

$$\omega_1=\frac{c_1A_2^2-a_1\kappa_1^2}{1-b_1\kappa_1}\tag{39}$$

and

$$\omega_2=\frac{(c_2A_1^2+d_2A_2^2)-a_2\kappa_2^2}{1-b_2\kappa_2}\tag{40}$$

subject to the constraints (12) and (13). Similarly, with the aid of (10) and (11) the width of the solitons can be written as

$$B=A_2\left[-\frac{c_1(1-b_1\kappa_1)}{2(a_1+a_1b_1\kappa_1-b_1^2\omega_1)}\right]^{\frac{1}{2}}\tag{41}$$

and

$$B=\left[-\frac{(1-b_2\kappa_2)(c_2A_1^2+d_2A_2^2)}{2(a_2+a_2b_2\kappa_2-b_2^2\omega_2)}\right]^{\frac{1}{2}}\tag{42}$$

which in turn introduce the constraints

$$c_1(1-b_1\kappa_1)\times(a_1+a_1b_1\kappa_1-b_1^2\omega_1)<0\tag{43}$$

$$\begin{aligned}&(1-b_2\kappa_2)(c_2A_1^2+d_2A_2^2)\\&\times(a_2+a_2b_2\kappa_2-b_2^2\omega_2)<0\end{aligned}\tag{44}$$

By equating the two values of the width B from (41) and (42) one retrieve again relation (31) with corresponding constraint (32). Therefore the dark 1-soliton solution to the cascaded system with STD (15)–(16) is given by

$$q(x,t)=A_1\tanh[B(x-vt)]e^{i(-\kappa_1x+\omega_1t+\theta_1)}\tag{45}$$

and

$$r(x,t)=A_2\tanh[B(x-vt)]e^{i(-\kappa_2x+\omega_2t+\theta_2)}\tag{46}$$

The constraint conditions that must remain valid in order for the dark solitons to exist are (12), (13), (32), (43) and (44).

3.1.3 Singular solitons

For singular solitons the assumption for the wave profile of the two components is

$$P_1=A_1\operatorname{csch}^{p_1}\tau\tag{47}$$

and

$$P_2=A_2\operatorname{csch}^{p_2}\tau\tag{48}$$

where τ is being defined as in (21), while the parameters A_1 , A_2 and B are considered herein as free parameters. Upon substituting (47) and (48) into (17) and (18) one get

$$\begin{aligned}&\left\{\begin{matrix}(b_1\omega_1\kappa_1-\omega_1-a_1\kappa_1^2)\\+p_1^2(a_1-b_1v)B^2\end{matrix}\right\}\operatorname{csch}^{p_1}\tau\\&+p_1(1+p_1)(a_1-b_1v)B^2\operatorname{csch}^{p_1+2}\tau\\&+c_1A_2^2\operatorname{csch}^{p_1+2p_2}\tau=0\end{aligned}\tag{49}$$

and

$$\begin{aligned}&\left\{\begin{matrix}(b_2\omega_2\kappa_2-\omega_2-a_2\kappa_2^2)\\+p_2^2(a_2-b_2v)B^2\end{matrix}\right\}\operatorname{csch}^{p_2}\tau\\&+p_2(1+p_2)(a_2-b_2v)B^2\operatorname{csch}^{p_2+2}\tau\\&+c_2A_1^2\operatorname{csch}^{p_2+2p_1}\tau+d_2A_2^2\operatorname{csch}^{3p_2}\tau=0\end{aligned}\tag{50}$$

Now, from (49), equating the exponent pair $(p_1 + 2p_2,\ p_1 + 2)$ while from (50), equating the exponent pairs $(p_2 + 2p_1,\ p_2 + 2)$ and $(3p_2,\ p_2 + 2)$ leads to the same value for p_1 and p_2 as in (24). Next, setting the coefficients of the linearly independent functions, $\operatorname{csch}^{p_j + l}\tau$ where $j = 1, 2$ and $l = 0, 2$, to zero in (49) and (50) leds to the wave numbers of the solitons being given by

$$\omega_1 = \frac{c_1 A_2^2 + 2a_1\kappa_1^2}{2(b_1\kappa_1 - 1)} \tag{51}$$

and

$$\omega_2 = \frac{(c_2 A_1^2 + d_2 A_2^2) + 2a_2\kappa_2^2}{2(b_2\kappa_2 - 1)} \tag{52}$$

subject to the constraints given in (12) and (13). In a similar manner, with the aid of (10) and (11), the width of the solitons can be retrieve as in (41) and (42) with corresponding constraints (43) and (44). Consequently one can retrieve relation (31) with corresponding constraint (32).

Therefore the dark 1-soliton solution to the cascaded system with STD (15)–(16) is given by

$$q(x,t) = A_1 \operatorname{csch}[B(x - vt)]e^{i(-\kappa_1 x + \omega_1 t + \theta_1)} \tag{53}$$

and

$$r(x,t) = A_2 \operatorname{csch}[B(x - vt)]e^{i(-\kappa_2 x + \omega_2 t + \theta_2)} \tag{54}$$

The constraint conditions that must remain valid in order for the singular solitons to exist are (12), (13), (32), (43) and (44).

3.2 Power law

Power law nonlinearity is typically observed in semiconductors for low powered nonlinearities. Here, $F(s) = s^n$, where n represents the strength of nonlinearity. In this case, stability issue dictates $0 < n < 2$ and also $n \neq 2$ in order to avoid self-focusing singularity [4, 5]. Thus, the system (1)–(2) is rewritten as

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + c_1 |r|^{2n} q = 0 \tag{55}$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + c_2 |q|^{2n} r + d_2 |r|^{2n} r = 0 \tag{56}$$

respectively. By substituting (3) and (4) into (55) and (56) the imaginary portions remain unchanged as in (8) and (9), while the resulting real parts obtained are

$$\begin{aligned} &a_1 \frac{\partial^2 P_1}{\partial x^2} + b_1 \frac{\partial^2 P_1}{\partial x \partial t} \\ &+ (b_1\omega_1\kappa_1 - \omega_1 - a_1\kappa_1^2)P_1 + c_1 P_2^{2n} P_1 = 0 \end{aligned} \tag{57}$$

$$\begin{aligned} &a_2 \frac{\partial^2 P_2}{\partial x^2} + b_2 \frac{\partial^2 P_2}{\partial x \partial t} \\ &+ (b_2\omega_2\kappa_2 - \omega_2 - a_2\kappa_2^2)P_2 \\ &+ c_2 P_1^{2n} P_2 + d_2 P_2^{2n+1} = 0 \end{aligned} \tag{58}$$

In order to analyze equations (57) and (58) the ansatz approach will be adopted depending on the type of solitons to be considered.

3.2.1 Bright solitons

For bright solitons, the starting hypothesis is the same as that of Kerr law nonlinearity given by (19) and (20) along with (21). After substitution, (57) and (58) reduce to

$$\begin{aligned} &\begin{Bmatrix} (b_1\omega_1\kappa_1 - \omega_1 - a_1\kappa_1^2) \\ + p_1^2(a_1 - b_1 v)B^2 \end{Bmatrix} \operatorname{sech}^{p_1}\tau \\ &- p_1(1 + p_1)(a_1 - b_1 v)B^2 \operatorname{sech}^{p_1 + 2}\tau \\ &+ c_1 A_2^{2n} \operatorname{sech}^{p_1 + 2np_2}\tau = 0 \end{aligned} \tag{59}$$

and

$$\begin{aligned} &\begin{Bmatrix} (b_2\omega_2\kappa_2 - \omega_2 - a_2\kappa_2^2) \\ + p_2^2(a_2 - b_2 v)B^2 \end{Bmatrix} \operatorname{sech}^{p_2}\tau \\ &- p_2(1 + p_2)(a_2 - b_2 v)B^2 \operatorname{sech}^{p_2 + 2}\tau \\ &+ c_2 A_1^{2n} \operatorname{sech}^{p_2 + 2np_1}\tau + d_2 A_2^{2n} \operatorname{sech}^{(2n+1)p_2}\tau = 0 \end{aligned} \tag{60}$$

Then, from (59), equating the exponent pair $(p_1 + 2np_2,\ p_1 + 2)$ while from (60) equating the exponent pairs from $(p_2 + 2, (2n+1)p_2,\ p_2 + 2np_1)$ leads to

$$p_1 = p_2 = \frac{1}{n} \tag{61}$$

Setting the coefficients of the linearly independent functions $\operatorname{sech}^{p_j + l}\tau$ where $j = 1, 2$ and $l = 0, 2$, to zero in (59) and (60) leads to the wave numbers of the solitons being given by

$$\omega_1 = \frac{c_1 A_2^2 - (n+1)a_1\kappa_1^2}{(n+1)(1 - b_1\kappa_1)} \tag{62}$$

and

$$\omega_2 = \frac{(c_2 A_1^2 + d_2 A_2^2) - (n+1) a_2 \kappa_2^2}{(n+1)(1-b_2\kappa_2)} \qquad (63)$$

subject to the constraints (12) and (13). Similarly, with the aid of (10) and (11) the width of the solitons can be retrieve as

$$B = nA_2^n \left[\frac{c_1(1-b_1\kappa_1)}{(n+1)(a_1 + a_1 b_1 \kappa_1 - b_1^2 \omega_1)} \right]^{\frac{1}{2}} \qquad (64)$$

and

$$B = n \left[\frac{(1-b_2\kappa_2)(c_2 A_1^{2n} + d_2 A_2^{2n})}{(n+1)(a_2 + a_2 b_2 \kappa_2 - b_2^2 \omega_2)} \right]^{\frac{1}{2}} \qquad (65)$$

which in turn introduce the constraints (29) and (30). By equating the two values of the width B from (64) and (65) one have

$$\frac{A_1}{A_2} = \left[\frac{c_1(1-b_1\kappa_1)(a_2 + a_2 b_2 \kappa_2 - b_2^2 \omega_2)}{c_2(1-b_2\kappa_2)(a_1 + a_1 b_1 \kappa_1 - b_1^2 \omega_1)} - \frac{d_2}{c_2} \right]^{\frac{1}{2n}} \qquad (66)$$

which is valid whenever the constraint (32) holds. Therefore the bright 1-soliton solution to the cascaded system with STD (57)–(58) is given by

$$q(x,t) = A_1 \operatorname{sech}^{\frac{1}{n}}[B(x-vt)] e^{i(-\kappa_1 x + \omega_1 t + \theta_1)} \qquad (67)$$

$$r(x,t) = A_2 \operatorname{sech}^{\frac{1}{n}}[B(x-vt)] e^{i(-\kappa_2 x + \omega_2 t + \theta_2)} \qquad (68)$$

The constraint conditions that must remain valid in order for this bright solitons to exist are (12), (13), (29), (30) and (32).

3.2.2 Dark solitons

For dark solitons the assumption for the wave profile of the two components remains intact as in (35) and (36). Thus, in this case the equations (57) and (58) modifies to

$$\begin{aligned}
&\left\{ \begin{aligned} &(b_1\omega_1\kappa_1 - \omega_1 - a_1\kappa_1^2) \\ &-2p_1^2(a_1 - b_1 v)B^2 \end{aligned} \right\} \tanh^{p_1}\tau \\
&+ p_1(p_1+1)(a_1 - b_1 v)B^2 \tanh^{p_1+2}\tau \\
&+ p_1(p_1-1)(a_1 - b_1 v)B^2 \tanh^{p_1-2}\tau \\
&+ c_1 A_2^{2n} \tanh^{p_1+2np_2}\tau = 0
\end{aligned} \qquad (69)$$

and

$$\begin{aligned}
&\left\{ \begin{aligned} &(b_2\omega_2\kappa_2 - \omega_2 - a_2\kappa_2^2) \\ &-2p_2^2(a_2 - b_2 v)B^2 \end{aligned} \right\} \tanh^{p_2}\tau \\
&+ p_2(p_2+1)(a_2 - b_2 v)B^2 \tanh^{p_2+2}\tau \\
&+ p_2(p_2-1)(a_2 - b_2 v)B^2 \tanh^{p_2-2}\tau \\
&+ c_2 A_1^{2n} \tanh^{p_2+2np_1}\tau \\
&+ d_2 A_2^{2n} \tanh^{(2n+1)p_2}\tau = 0
\end{aligned} \qquad (70)$$

From (69), equating the exponent pair $(p_1 + 2np_2,\ p_1 + 2)$ and from (70), equating the exponents $((2n+1)p_2,\ 2np_1 + p_2,\ p_2 + 2)$ pairwise, leads to (61). Then setting the coefficient of the stand alone linearly independent functions $\tanh^{p_j - 2}\tau$ to zero in (69) and (70) leads to (24). Then, from (24) and (61), one can conclude that $n = 1$, mening that for the cascade system (1)-(2) with power law nonlinearity, dark solitons will exist provided the power law nonlinearity reduces to Kerr law nonlinearity. Consequently, all the results of dark solitons for Kerr law nonlinearity given by (39)–(46) will follow.

3.2.3 Singular solitons

To derive singular soliton solutions from (55)-(56) the assumption for the wave profile of the two components remain as in (47)-(48) with τ being defined as in (21). Upon substituting (47) and (48) into (57) and (58) one obtains

$$\begin{aligned}
&\left\{ \begin{aligned} &(b_1\omega_1\kappa_1 - \omega_1 - a_1\kappa_1^2) \\ &+ p_1^2(a_1 - b_1 v)B^2 \end{aligned} \right\} \operatorname{csch}^{p_1}\tau \\
&+ p_1(1+p_1)(a_1 - b_1 v)B^2 \operatorname{csch}^{p_1+2}\tau \\
&+ c_1 A_2^{2n} \operatorname{csch}^{p_1+2np_2}\tau = 0
\end{aligned} \qquad (71)$$

and

$$\begin{aligned}
&\left\{ \begin{aligned} &(b_2\omega_2\kappa_2 - \omega_2 - a_2\kappa_2^2) \\ &+ p_2^2(a_2 - b_2 v)B^2 \end{aligned} \right\} \operatorname{csch}^{p_2}\tau \\
&+ p_2(1+p_2)(a_2 - b_2 v)B^2 \operatorname{csch}^{p_2+2}\tau \\
&+ c_2 A_1^{2n} \operatorname{csch}^{p_2+2np_1}\tau \\
&+ d_2 A_2^{2n} \operatorname{csch}^{(2n+1)p_2}\tau = 0
\end{aligned} \qquad (72)$$

Now, from (71), equating the exponent pair $(p_1 + 2np_2,\ p_1 + 2)$ while from (72), equating the exponent pairs $(p_2 + 2np_1,\ p_2 + 2)$ and $((2n+1)p_2,\ p_2 + 2)$ leads to the same value for p_1 and p_2 as in (61). Next, setting the coefficients of the linearly independent functions, $\operatorname{csch}^{p_j+l}\tau$ where $j=1, 2$ and $l=0, 2$, to zero in (71) and (72) leds to the wave numbers of the solitons being given by

$$\omega_1 = \frac{c_1 A_2^{2n} + (n+1)a_1\kappa_1^2}{(n+1)(b_1\kappa_1 - 1)} \tag{73}$$

and

$$\omega_2 = \frac{(c_2 A_1^{2n} + d_2 A_2^{2n}) + (n+1)a_2\kappa_2^2}{(n+1)(b_2\kappa_2 - 1)} \tag{74}$$

subject to the constraints (12) and (13). Similarly, with the aid of (10) and (11) the width of the solitons can be retrieve as

$$B = nA_2^n\left[-\frac{c_1(1-b_1\kappa_1)}{(n+1)(a_1 + a_1 b_1\kappa_1 - b_1^2\omega_1)}\right]^{\frac{1}{2}} \tag{75}$$

and

$$B = n\left[-\frac{(1-b_2\kappa_2)(c_2 A_1^{2n} + d_2 A_2^{2n})}{(n+1)(a_2 + a_2 b_2\kappa_2 - b_2^2\omega_2)}\right]^{\frac{1}{2}} \tag{76}$$

which in turn introduce the constraints (43) and (44). By equating the two values of the width B from (75) and (76) one retrieve again relation (66) with constraint as in (32). Therefore the singular 1-soliton solution to the cascaded system with STD (57)–(58) is given by

$$q(x,t) = A_1 \operatorname{csch}^{\frac{1}{n}}[B(x - vt)]e^{i(-\kappa_1 x + \omega_1 t + \theta_1)} \tag{77}$$

and

$$r(x,t) = A_2 \operatorname{csch}^{\frac{1}{n}}[B(x - vt)]e^{i(-\kappa_2 x + \omega_2 t + \theta_2)} \tag{78}$$

The constraint conditions that must remain valid in order for the singular solitons to exist are (12), (13), (32), (43) and (44).

4. Conclusions

This paper studied optical solitons in cascaded system where STD term is included in addition to GVD. STD provides well-posedness to the problem as indicated during 2012. The results are thus a generalized version of results that was reported earlier [4]. Bright, dark and singular 1-soliton solutions are reported in this paper along with the respective constraint conditions. These conditions provide the guarantee for the solitons to exist.

There are several aspects to the future of this paper. Later, perturbation terms will be added to this paper and thus soliton solutions to cascaded system with perturbation terms will be reported. Additionally, numerical simulations will be obtained to illustrate the mathematical mechanism. These form a tip of the iceberg.

Acknowledgments

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University under grant number (64-130-35-HiCi). The authors therefore thankfully acknowledge this technical and financial support of KAU.

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