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Overview of the superradiant phase transition: the Dicke model[♣]

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The real interaction between matter and electromagnetic radiation is too complicated for a complete theoretical investigation. Nevertheless, in some cases the problem admits an amazing simplification which allows one to consider interesting phenomena in the framework of rather simple models having even exact solutions. A model, which describes in the dipolar approximation the interaction of N two-level atoms with a quantized radiation field in an ideal cavity with volume V, bears the name of Dicke. This model is of key importance for describing dynamical, collective and coherent effects in quantum optics. Since 1974, when Hepp and Lieb rigorously proved that the Dicke model exhibits a second order phase transition from the normal to a superradiant phase, its thermodynamic properties have been studied in detail in the context of critical phenomena and solid state physics. Quite recently, a new aspect emerged when it was realized that the quantum phase transition of the model is relevant to quantum information and quantum computing. Various physical approximations have been extensively debated in the above mentioned fields of research. Here, an attempt is made to review in a rigorous manner the thermodynamic properties of the original Dicke model and its different generalizations. Some new results concerning relations between different indicators of criticality are presented as well.

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"*Mathematics and physics are different enterprises: physics is looking for laws of nature, mathematics is trying to invent the structures and prove the theorems of mathematics. Of course these structures are not invented out of thin air but are linked, among other things, to physics"*.
P.D. Lax [1].

1. Introduction

Since 1954, when Dicke suggested in his classical paper [2] the model which now bears his name, there has been an increasing activity in diverse fields of physics bearing a relation to the Dicke model (DM), such as critical phenomena, chaos, mesoscopics (level statistics, scaling), among others, see the review [3]. The investigation of the model in the context of quantum statistical mechanics was initiated by the seminal paper by Hepp and Lieb [4].They demonstrated that DM exhibits a second order phase transition from the normal to a superradiant phase. This is a striking example of when a macroscopic many-particle quantum phenomenon is predicted in a manner which is mathematically rigorous and exact. Up to now, this subject has been an area of exciting theoretical and mathematical research, which appear as a remarkable illustration of Lax's statement. The observation of the superradiant phase transition (SPT) still remains a challenge for experiments. The hard problem is to provide a practical system suitable to realize the DM experimentally, see e.g. [5, 6] and references therein. In quantum electrodynamics, DM describes light-matter interaction in a photon cavity. With respect to the transport properties, candidates for experimental systems are arrays of excitonic quantum dots and electrons in quantum dots interacting with single phonon modes (see, e.g. the review [3]). We should mention also a recent observation of wave - matter amplification due to Raman superradiant light scattering in the Bose condensate of atoms in traps [7, 8]. It makes the DM relevant to this case [9]. Recently, a combination of a superconducting quantum interference device (SQUID) with a nanomechanical resonator (NAMR) has been advocated [10] as an experimental

[♣] Paper presented at the International School on Condensed Matter Physics, Varna, Bulgaria, September 2006 scheme for testing the theoretical predictions of the model. It is remarkable that the same combination of devices can be regarded as an attractive solid state candidate (see the references in [10]) for processing quantum entanglement [11] in a system that would implement quantum computing [12]. Entanglement is a purely quantum mechanical feature which has been extensively studied in the last decades. Recent research confirms that entanglement really exists on a macroscopic level and may be related to known thermodynamic properties of quantum many-body systems. The DM, being exactly solvable, would play the role of a key model when looking for entanglement as a thermodynamic property. Here, we summarise the results of our investigations in the field of phase transitions of DM. In particular, we discuss indicators of criticality such as: order parameters, different susceptibilities, cross-correlations for both electromagnetic and atomic constituents of the superradiant state, etc. Our consideration of the thermodynamic properties of the model may be helpful in a further quest for entanglement detection.

2. The single mode model

The Dicke model of superradiance describes the interaction in a dipolar approximation of N two-level atoms with one-mode of the electromagnetic field in an ideal cavity of volume V [2]. It is commonly used to illustrate collective effects in the atom-light interaction, leading to the concept of superradiance, when the atomic ensemble spontaneously emits electromagnetic waves with an intensity proportional to N^2, rather than to N as one would expect if the atoms were radiating incoherently [3]. A two-level atom (called also spin 1/2) in the DM is described in terms of the Pauli matrices σ^+ and σ^- acting on a two dimensional complex Hilbert space C^2 and satisfying the anticommutation relations $\{\sigma^+,\sigma^+\}=2(\sigma^+)^2=\{\sigma^-,\sigma^-\}=0$, $\{\sigma^+,\sigma^-\}=1$. The single mode of the electromagnetic field is described by the creation and annihilation operators of a harmonic oscillator a^+,a acting on the one-mode Fock space F_B^1 and satisfying the commutation relations $[a^+,a]=1, [a^+,a^+]=[a,a]=0$. The Hilbert space for the composite system in the single mode case is $F=F_B^1\otimes\left(C^2\right)^{\otimes N}$. The Hamiltonian of the Dicke model has the form ($\hbar=c=1$),

$$H=\omega a^+a+\varepsilon S^z+\frac{\lambda}{N^{1/2}}\left(a^+J^-+aJ^+\right). \qquad (1)$$

Here $0<\omega<\infty$ is the frequency of one mode of the electromagnetic field, $\varepsilon\in R$ is the atomic level splitting, $0<\lambda<\infty$ is the atom-field coupling, $S^z=\frac{1}{2}\sum_{i=1}^{N}\sigma_i^z$, $J^{\pm}=\frac{1}{2}\sum_{i=1}^{N}\left(\sigma_i^{\pm}+\mu\sigma_i^{\mp}\right)$.

The factor $N^{-1/2}$ comes from the original dipole coupling strength which is proportional to $V^{-1/2}$; as far as one considers a fixed density $\rho=N/V$, the difference amounts to a coupling constant renormalization. Only parameters $\mu=1$ and $\mu=0$ have a physical meaning. The latter case is known as the rotating wave approximation (RWA).

The Hamiltonian H is a self-adjoint operator on $D(T)$, the domain of $T=\omega a^+a$. It is bounded from below and has a purely discrete spectrum with finite multiplicity. The operator $\exp(-\beta H)$ is of *Trace*-class for all inverse temperatures $\beta>0$ and the limit $N\to\infty$ of the thermodynamic potential per spin $f_N[H]=-(\beta N)^{-1}\ln\operatorname{Tr}\exp(-\beta H)$ exists (see, e.g. [13]). The large-N asymptotic behaviour of the eigenvalues of H has been studied by exact quantum mechanical methods [14]. It provided a hint that the model exhibits a "phase transition" governed by λ to a ground state with a macroscopic number of photons and spontaneously excited atoms [14]. It is known that in some cases DM is integrable by the Bethe anzatz techniques for an arbitrary number of atoms, reducing the problem to an algebraic equation from which it is far from being trivial to extract information for the general eigenvalues and eigenfunctions [15].

3. The thermodynamics of the model

The equilibrium statistical mechanical properties of DM can be studied exactly and rigorously by different methods. The first consideration proposed in [4] was rather tedious and limited to the case of one-mode RWA. For a finite number of modes and beyond the RWA approximation, more convenient and transparent are the *Coherent State Method* (CSM) [16] and the *Approximating Hamiltonian Method* (AHM) [17]. In the AHM, the model Hamiltonian H is simplified by replacing some spin operator constructions by *c*-numbers. The resulting Hamiltonian is called the approximating Hamiltonian $H^{appr}(C)$ if, under a proper choice of the parameters C, it can be proved to be equivalent to the initial one, in the sense that both Hamiltonians generate the same thermodynamic behaviour in the thermodynamic limit $N/V=const$, $N\to\infty$. Several comments are in order here. The Dicke model has been studied by different approximate methods which have yielded exact results. For example Wang and Hioe [18] first used the CSM, however without complete mathematical reasoning. The proof was given later by Hepp and Lieb [16]. Similarly, there is the status of the approach by Bogoliubov, Zubarev and Tzerkovnikov (BZT) [19], used for the Dicke model by Giberd [20]. A comment on the foundations and the lack of mathematical rigour of the BZT approach can be found in [21]. The wisdom is that some methods, which lack mathematically rigorous foundations, may yield exact results for particular models, but in other cases may not, see e.g. [22]. The model (1) exhibits a second order phase transition driven by the temperature from a normal phase to a superradiant phase, with a macroscopically occupied boson mode and highly correlated atomic states that possess the ability to superradiate [4, 16, 17]. For parameters $\Lambda^2 := (1+\mu)^2 \lambda^2/\omega$ and ε obeying the condition $\Lambda^2 < |\varepsilon|$, no phase transition occurs at any temperature, whereas for $\Lambda^2 > |\varepsilon|$ there exists a finite inverse critical temperature β_c given by $\beta_c = \frac{2}{|\varepsilon|}\tanh^{-1}\left(\frac{|\varepsilon|}{\Lambda^2}\right)$. There is another phase transition of second order driven by the parameter λ, which the model exhibits at the point $T=0$, $\Lambda^2 = |\varepsilon|$. This kind of phase transition is driven by the quantum fluctuations associated with Heisenberg's uncertainty principle, rather than by the thermal motion. Quite recently, a renewed interest in this issue has appeared due to studies of quantum entanglement [3, 23]. The quantum SPT at the critical point $\lambda_c = \sqrt{|\varepsilon|\omega}/|1+\mu|$ manifests itself, among other phenomena, in the entanglement between the atomic ensemble and the field mode. It manifests itself also as a matter-wave grating which has been observed experimentally [7-9]. In the normal phase

$$\lim_{N\to\infty} N^{-1}\left\langle a^+ a\right\rangle_H = 0,$$

where

$$\langle\ldots\rangle_H := Tr\left[\ldots\exp(-\beta H)\right]\left[Tr\exp(-\beta H)\right]^{-1}.$$

In the superradiant phase, there is a macroscopic occupation of the boson mode:

$$\lim_{N\to\infty} N^{-1}\left\langle a^+ a\right\rangle_H = \frac{\lambda^2}{\omega^2}|C(\mu)|^2,$$

where $C \neq 0$ is a non-trivial solution of the self-consistent equation:

$$|C| = \frac{\Lambda^2 |C|}{E(C)}\tanh\frac{\beta E(C)}{2}$$

and

$$E(C) := \left[\varepsilon^2 + \left(4\lambda^4/\omega^2\right)(1+\mu)^2 |C|^2\right]^{\frac{1}{2}}..$$

The self-consistent equation for C indicates that the SPT is of mean field type. There exists a physical restriction on the parameters of the DM imposed by the Thomas-Reiche-Kühn sum rule that forbids the SPT [24]. The problem is closely related to the various approximations (e.g. with the usually neglected A^2-term) in the derivation of the original DM (1), see e.g. [25, 26].

4. Finite and infinite numbers of photon modes

No principal difficulties are encountered in extending the above consideration to the more general M - mode case $(1 \le M \le M_0 < \infty)$, with k-dependent model constants, k being the length of the wave vector. and the Hamiltonian:

$$\begin{aligned} H_M &= \sum_{k=1}^{M}\omega(k)\, a_k^+ a_k + \varepsilon S^z + \\ &+ N^{-1/2}\sum_{k=1}^{M}\lambda(k)\left(a_k^+ J_k^- + a_k J_k^+\right), \end{aligned} \tag{2}$$

where $J_k^{\pm} = \frac{1}{2}\sum_{i=1}^{N}\left(\sigma_i^{\pm} + \mu_k \sigma_i^{\mp}\right)$.

Now the state space is $F = \otimes_{k=1}^{M} F_B^k \otimes (C^2)^{\otimes N}$.

The main conclusion is that there is effectively only one combination of the coupling constants in the model, $\Lambda_M^2 := \sum_{k=1}^{M}(1+\mu_k)^2\lambda^2(k)/\omega(k)$, which governs the phase transition. The thermodynamic behaviour of the M-mode DM can be obtained by replacing Λ^2 with Λ_M^2 and $E(C)$ with

$$E_M(C) := \left\{\varepsilon^2 + 4\left[\sum_{k=1}^{M}\frac{\lambda^2(k)}{\omega(k)}\left(1+\mu_k\right)|C|\right]^2\right\}^{1/2}$$

in the one-mode results [4, 16, 17].

The generalization of the above results to the case of an infinite number of photon modes $M \sim N \to \infty$ is possible in two ways [26, 27, 28]. The first one seems to be straightforward, provided the condition $\sum_{k=1}^{\infty}\lambda^2(k)/\omega(k) \le R_1 < \infty$ is fulfilled. In fact the majorization technique of the AHM requires an additional condition. While the result is certainly correct in the mathematical sense, this condition has an obvious physical drawback: it ignores the proportionality of the number of modes of the field to the volume V of the cavity. In order to avoid this difficulty, in the genuine model (2) the replacement $M \to N$ is attended to by the substitution

$\lambda(k) \to \tilde{\lambda}(k) N^{-1/2}$ (the additional dependence on $N^{-1/2}$ is taken for thermodynamic reasons), and then the condition $N^{-1}\sum_{k=1}^{\infty} \tilde{\lambda}^2(k)/\omega(k) \le \tilde{R} < \infty$ is imposed. For each finite integer N, the state space of the system is the N-fold completed tensor product $\otimes_{k=1}^{N} F_B^k \otimes (C^2)^{\otimes N}$. This model exhibits a phase transition at a critical temperature which coincides with the critical temperature of (1). However, in contrast to the case of a finite number of modes, there is no Bose-condensation of the electromagnetic modes. The phase transition here reflects only a deviation from the ideal Bose-distribution below β_c^{-1} [27].

5. Asymptotic convergence of thermodynamic averages

In a strict sense, thermodynamic equivalence of two Hamiltonians means not only convergence in the thermodynamic limit of the corresponding thermodynamic potentials, but also convergence of the corresponding Gibbs states. Technically speaking, thermodynamic averages can be evaluated as functions of derivatives of the thermodynamic potentials with respect to appropriate coupling parameters in the Hamiltonians (for finite N, these potentials are analytic functions). The proof of the convergence of the derivatives of a sequence of functions to a limit derivative requires additional mathematical efforts, and as a rule is true only under certain conditions. In this case, the following theorem due to Griffiths - Fisher (see, e.g. [29, 30, 31]) is useful. Let $\{f_n(x)\}$, $x \in I \subset R$, be a sequence of convex functions which converges point wise to $f_\infty(x)$ as $n \to \infty$. Then the left, $f'(x-0)$, and the right, $f'(x+0)$, derivatives at any point $x \in I$ obey the inequalities

$$f'_\infty(x-0) \le \liminf_{n\to\infty} f'_n(x-0) \le \limsup_{n\to\infty} f'_n(x+0) \le f'_\infty(x+0).$$

If $\{f_n(x)\}$ and $f'_\infty(x)$ are differentiable at a point $x_0 \in I$, then $\lim_{n\to\infty} f'_n(x_0) = f'_\infty(x_0)$.

This theorem is useful in proving the asymptotic closeness of certain average values in the model and approximating system. For applications, see Section 6. Another example of a mathematical result that has shed some light on the problem is a theorem originally due to Hadamard and Kolmogorov, which we give in a slightly different version [29, 30, 31]: Let $\Delta_n(x), n = 1,2,...,$ be a sequence of functions that are continuously differentiable in the interval $I = [a,b] \in R$ and have second derivatives $\Delta''_n\ (x+0)$ to the right of each point $s \in [a,b)$. Suppose that for all $x \in I$ $|\Delta_n(x)| \le \varepsilon_n(I) \to 0$, as $n \to \infty$, and that there exists a fixed positive number $D(I)$ such that one of the following two conditions holds for all $x \in [a,b)$: *(i)* $\Delta''_n\ (x+0) \le D(I)$ or *(ii)* $\Delta''_n\ (x+0) \ge -D(I)$. Then, for all x and n satisfying the inequalities $a + l_n \le x \le b - l_n$, where $l_n = 2[\varepsilon_n(I) D(I)]^{1/2}$, the following upper bound of the first derivative $\Delta'_n(x)$ holds:

$$|\Delta'_n(x)| \le l_n \to 0, \text{ as } n \to \infty.$$

How useful these theorems are can be seen in [17, 26, 30]. In the case under consideration, we can set $\Delta_N(h) = f_N[H(h)] - f_N[H^{appr}(h)]$, where

$$H(h) = H + hNQ \quad \text{and} \quad H^{appr}(h) = H^{appr} + hNQ.$$

The operator Q may correspond to S^z/N, $J^{\pm}/N$ or to different powers of these operators. The auxiliary field $h \in R$ or C. If one can establish that $\Delta_N(h)$ obeys the conditions of the theorem, one can estimate the closeness of the first derivatives of the free energy densities. However, for explicit calculation of the Gibbs average value of Q, this operator must be such that the function $f_N[H^{appr}(h)]$ is explicitly known.

We can obtain some important relations between the Gibbs mean values of operators related to the field and atomic constituents of the DM, by introducing sources for photons in the Hamiltonian (1):

$H(\nu) = H - \sqrt{N}(\nu a^+ + \nu^* a)$, where $\nu \in C$. Then $H(\nu)$ can be identically represented in the form

$$\tilde{H}(\nu) = \omega b^+ b + \varepsilon S^z + \frac{\lambda}{N^{1/2}}(b^+ J^- + b J^+) + \\ + \frac{\lambda}{\omega}(J^+ \nu + J^- \nu^*) - \frac{N}{\omega}|\nu|^2,$$

where new photon operators have been introduced as $b^+ = a^+ - \frac{\sqrt{N}}{\omega}\nu^+$, $(b^+)^+ = b$. Since b^+, b and a^+, a are related by a unitary transformation, we have $f_N[H(\nu)] = f_N[\tilde{H}(\nu)]$. From this relation, one immediately obtains the equality of the derivatives

$$\frac{\partial^{n+m}}{\partial \nu^n \partial(\nu^*)^m} f_N[H(\nu)] = \frac{\partial^{n+m}}{\partial \nu^n \partial(\nu^*)^m} f_N[\tilde{H}(\nu)].$$

If $n = 1, m = 0$, after setting $\nu = 0$ one obtains

$$\left\langle \frac{a}{\sqrt{N}} \right\rangle_H = -\frac{\lambda}{\omega}\left\langle \frac{J}{N} \right\rangle_H$$

and a similar equality for the hermitian conjugate. If $n = m = 1$, after setting $\nu = 0$ one obtains

$$\left(\frac{a}{\sqrt{N}}, \frac{a}{\sqrt{N}}\right)_H = \frac{\lambda^2}{\omega^2}\left(\frac{J}{N}, \frac{J}{N}\right)_H + \frac{1}{\beta\omega N},$$

where the Bogoliubov-Duhamel inner product $(A, A)_H = \frac{1}{\beta} \mathrm{Tr}\, \mathrm{e}^{-\beta H} \int_0^\beta d\tau \, \mathrm{Tr}\left[\mathrm{e}^{-(\beta-\tau)H} A^+ \mathrm{e}^{-\beta H} A\right]$ has been introduced. The above result implies an important relationship between the susceptibilities χ of the electromagnetic field and the atomic subsystem: $\chi_a[H] = \chi_a[\omega a^+ a] + \chi_J[H]$. Here, the subscripts a and J indicate that the second order derivatives are taken over the external fields $\nu \in C$, after the terms $\nu^* a, \nu a^+$ or $\nu^* J^-, \nu J^+$ have been added to the Hamiltonian in the bracket. In the above equality, the external fields are set equal to zero after the derivation, to ensure the comparison of the response properties of the field and atomic constituents of the system in the same thermal state. It is important that Eqs. (8), (9) and (10) are valid for every finite N.

6. The equivalence of DM to the LMG model

In the literature, there exist different statements of the equivalence of DM to models with direct spin-spin interaction, for comments see e.g. [22] and below. What has been proved [17] is that the Hamiltonian (2) is thermodynamically equivalent *on the level of the free energy densities* to the Hamiltonian:

$$H_S = \varepsilon S^z - \frac{1}{N}\sum_{k=1}^{M} \frac{\lambda^2(k)}{\omega(k)} J_k^+ J_k^- . \tag{3}$$

This result has been formalized as a mathematical theorem for a much larger class of models of matter interacting with boson fields, a particular case of which is the DM, see Theorem 4.1 in [29] (and also in [30]). For the free energy densities $f_N[H_M] := -(\beta N)^{-1} \ln \mathrm{Tr} \exp(-\beta H_M)$ and $f_N[H_S] := -(\beta N)^{-1} \ln \mathrm{Tr} \exp(-\beta H_S)$ the following estimates have been obtained by the *AHM*:

$$-\delta_N^H \le f_N[H_M] - f_N[H_S] \le \delta_N^B, \tag{4}$$

where $\delta_N^H = O(N^{-1/2})$ and $\delta_N^B = O(N^{-1}\ln N)$. The N-dependence in the above bounds holds for all values of the temperature and the model parameters. This means that for studying the thermodynamics of the systems of two-level atoms at any ε, $\lambda(k)$, $\omega(k)$ and in the whole temperature interval $0 \le T < \infty$, the effective Hamiltonian H_S can be used. The statement holds in the thermodynamic limit $N/V = const$, $N \to \infty$. If $\{\mu_k = 1, k = 1, ..., M\}$ or if $\{\mu_k = 0,\ k = 1, ..., M\}$, then the M-mode DM is thermodynamically equivalent to the infinitely long-range Ising model in a transverse field

$$H_S^X = \varepsilon S^z - \frac{1}{N}\Lambda_M^2(1)\left(S^x\right)^2 , \tag{4a}$$

or to the infinitely long-range isotropic XY model in a transverse field,

$$H_S^{XY} = \varepsilon S^z - \frac{1}{N}\Lambda_M^2(0)\left[(S^x)^2 + (S^y)^2\right], \tag{4b}$$

respectively.

Recently, the entanglement properties of systems undergoing quantum phase transitions have been studied in the framework of another model – the Lipkin-Meshkov-Glick model (LMGM) [32]. It has been conjectured in the literature that in some cases the LMGM can be put into a one-to-one correspondence with the DM. This statement has not been proved and its correctness is somewhat doubtful due to the fact that it is based on a limiting procedure - the thermodynamic limit.

To illustrate this point, consider the finite anisotropic LMG model with the Hamiltonian

$$H_N = \varepsilon S^z - \frac{g}{2N}\left[\left(S^x\right)^2 + \gamma\left(S^y\right)^2\right],$$

where we have set $S^\alpha = \sum_{i=1}^N \sigma_i^\alpha$, $\alpha = x, y, z$ and $g > 0$, $\gamma > 0$, and $\varepsilon \in R$ are parameters. This model belongs to the class of models described by Eq. (3), and in this sense is equivalent to the DM after the appropriate choice of the model constants, see Section 4. On the other hand, the Hamiltonian H_N is thermodynamically equivalent *on the level of free energy densities* to the one-particle approximating Hamiltonian

$$H_N^{appr}(a,b) = \varepsilon S^z - g\left(aS^x + \gamma b S^y\right) + \frac{g}{2}N(a^2 + \gamma b^2)$$

where the variational parameters a and b have to be determined from the absolute minimum condition for the approximating free energy density:

$$\begin{aligned} \min_{a,b\in R} f_N[H_N^{appr}(a,b)] &= f_N[H_N^{appr}(\bar{a},\bar{b})] = \\ &= -\beta^{-1}\ln 2\cosh\beta\sqrt{\varepsilon^2 + g^2(\bar{a}^2 + \gamma^2\bar{b}^2)} + \\ &+ \frac{1}{2}g(\bar{a}^2 + \gamma\bar{b}^2) \quad . \end{aligned} \tag{5}$$

The values $\bar{a}$ and $\bar{b}$, at which the absolute minimum of the approximating free energy density is reached, are solutions of the equations for extremum of $f_N[H_N^{appr}(a,b)]$ (known as self-consistency equations):

$$\left\langle \frac{S^x}{N} \right\rangle_{H_N^{appr}(a,b)} = a, \quad \left\langle \frac{S^y}{N} \right\rangle_{H_N^{appr}(a,b)} = b.$$

In other words, with the aid of the AHM one can prove that

$$\lim_{N\to\infty} f_N\left[H_N\right] = \\ = \lim_{N\to\infty} \min_{a,b\in R} f_N\left[H_N^{appr}(a,b)\right]. \tag{6}$$

Since the approximating free energy density (5) does not depend on N, the limit in the r.h.s. of Eq. (6) is trivially taken.

By differentiation of $f_N[H_N^{appr}(a,b)]$, we obtain the set of explicit equations

$$ga \frac{\tanh \beta\sqrt{\varepsilon^2 + g^2(a^2+\gamma^2 b^2)}}{\sqrt{\varepsilon^2 + g^2(a^2+\gamma^2 b^2)}} = a \ , \tag{7a}$$

$$g\gamma b \ \frac{\tanh \beta\sqrt{\varepsilon^2 + g^2(a^2+\gamma^2 b^2)}}{\sqrt{\varepsilon^2 + g^2(a^2+\gamma^2 b^2)}} = b \ , \tag{7b}$$

which are also independent of N. Obviously, they always have a trivial solution $\bar{a}=0$, $\bar{b}=0$, for which the approximating free energy density equals $f_N[H_N^{appr}(0,0)] = -\beta^{-1}\ln 2\cosh\beta\varepsilon$. A nontrivial solution of Eqs. (7) with both $\bar{a}\neq 0$ and $\bar{b}\neq 0$ exists only in the isotropic case $\gamma=1$. One can prove that when $\gamma<1$ the absolute minimum is reached at a non-trivial solution with $\bar{a}\neq 0$, $b=0$, and when $\gamma>1$ the nontrivial solution is $\bar{a}=0$, $\bar{b}\neq 0$. This is most readily seen in the zero-temperature case. Then, by taking the limit $\beta\to\infty$ in (5), we obtain for the ground state energy per spin the simple expression

$$\lim_{\beta\to\infty} f_N[H_N^{appr}(\bar{a},\bar{b})] \equiv e_0(\bar{a},\bar{b}) = \\ = -\sqrt{\varepsilon^2 + g^2(\bar{a}^2+\gamma^2\bar{b}^2)} + \frac{1}{2}g(\bar{a}^2+\gamma\bar{b}^2),$$

and equations (7a) and (7b) simplify to

$$a\sqrt{\varepsilon^2 + g^2(a^2+\gamma^2 b^2)} = ga \ , \tag{8a}$$

$$b\sqrt{\varepsilon^2 + g^2(a^2+\gamma^2 b^2)} = \gamma g b \, . \tag{8b}$$

In the asymmetric case $\gamma\neq 1$, the non-trivial solution $\bar{a}\neq 0$, $\bar{b}=0$ exists when $g>g_c^a\equiv|\varepsilon|$ and reads $\bar{a}=\pm\sqrt{1-(\varepsilon/g)^2}$. Then, the ground state energy density is $e_0(\bar{a},0) = -\frac{1}{2}|\varepsilon|(g/g_c^a + g_c^a/g) < e_0(0,0)$.

The nontrivial solution $\bar{a}=0$, $\bar{b}\neq 0$ exists when $g>g_c^b\equiv|\varepsilon|/\gamma$ and equals $\bar{b}=\pm\sqrt{1-(\varepsilon/\gamma g)^2}$. The corresponding ground state energy density is $e_0(0,\bar{b}) = -\frac{1}{2}|\varepsilon|(g/g_c^b + g_c^b/g) < e_0(0,0)$.

If $0<\gamma<1$, then $g_c^a < g_c^b$ and $g/g_c^a > g/g_c^b$. Therefore, for all $g>g_c^a\equiv|\varepsilon|$ the absolute minimum of the free energy density is attained at the nontrivial solution $\bar{a}\neq 0$, $\bar{b}=0$, and from the self-consistency equations it follows that

$$\lim_{N\to\infty} \left\langle \frac{S^x}{N} \right\rangle_{H_N} = \left\langle \frac{S^x}{N} \right\rangle_{H_N^{appr}(\bar{a},0)} = \pm\sqrt{1-(\varepsilon/g)^2} \tag{9a}$$

where the + or − sign corresponds to an appropriate symmetry broken phase, whereas

$$\lim_{N\to\infty} \left\langle \frac{S^y}{N} \right\rangle_{H_N} = \left\langle \frac{S^y}{N} \right\rangle_{H_N^{appr}(\bar{a},0)} = 0\, . \tag{9b}$$

Similarly, after diagonalization of $H_N^{appr}(\bar{a},0)$ when $g>g_c^a\equiv|\varepsilon|$, one can calculate

$$\left\langle \frac{S^z}{N} \right\rangle_{H_N^{appr}(\bar{a},0)} = \frac{-\varepsilon}{\sqrt{\varepsilon^2+g^2\bar{a}^2}} \tanh\beta\sqrt{\varepsilon^2+g^2\bar{a}^2}$$

Hence, in the ground state one, obtains

$$\lim_{N\to\infty} \left\langle \frac{S^z}{N} \right\rangle_{H_N} = \left\langle \frac{S^z}{N} \right\rangle_{H_N^{appr}(\bar{a},0)} = \frac{-\varepsilon}{g}\, .$$

However, one needs a special proof for the asymptotic convergence of the average values of the normalized total spin projections to the corresponding averages for the approximating system. The same is true for the average values of the normalized squared total spin projections. Note that the calculation of the latter values for the approximating system is straightforward and yields ($\gamma<1$):

$$\left\langle \left(\frac{S^x}{N} \right)^2 \right\rangle_{H_N^{appr}(\bar{a},0)} = 1 - \left(\frac{\varepsilon}{g} \right)^2 \left(1 - \frac{1}{N} \right)$$

$$\left\langle \left(\frac{S^y}{N} \right)^2 \right\rangle_{H_N^{appr}(\bar{a},0)} = \frac{1}{N}$$

$$\left\langle \left(\frac{S^z}{N} \right)^2 \right\rangle_{H_N^{appr}(\bar{a},0)} = \left(\frac{\varepsilon}{g} \right)^2 \left(1 - \frac{1}{N} \right) + \frac{1}{N}$$

By using the convergence (6) and the Griffiths-Fisher theorem, applied to the derivatives with respect to ε, g and γ of the free energy densities for the model system, $f_N(\varepsilon, g, \gamma) := f_N[H_N]$, and the approximating one, see Eq. (5), one can prove that the Gibbs mean values of the operators S^z/N, $(S^x/N)^2$ and $(S^y/N)^2$, taken with the Hamiltonians H_N and H_N^{appr}, coincide in the thermodynamic limit. Note that the above obtained mean values satisfy the identity

$$\frac{1}{N^2} \left\langle \left(S^x \right)^2 + \left(S^y \right)^2 + \left(S^z \right)^2 \right\rangle_{H_N} = 1 + \frac{2}{N}.$$

The precise meaning of the equalities in (9) needs some explanation. In order to calculate the mean value of S^x, or S^y, one adds to the model Hamiltonian H_N the term $\pm hS^x$ or $\pm hS^y$, respectively. Then, the derivative of the free energy density with respect to the auxiliary field h yields the corresponding mean value. Since this procedure changes the symmetry of the model Hamiltonian H_N, and does not change the symmetry of H_N^{appr}, the so obtained mean values are equivalent only in the sense of Bogoliubov's quasi-averages, i.e. only when the auxiliary field h is set to zero after the thermodynamic limit. Only in this sense do Eqs. (9) hold true. Note that the usual mean values of S^x and S^y for the model system with the Hamiltonian H_N are identically equal to zero, due to symmetry. As we have shown, this is not the case for the approximating Hamiltonian H_N^{appr}. In the thermodynamic limit, the above results coincide with those obtained in [33] by a different method. With regard to the finite-size corrections, the following comment is in order. Due to the one-particle character of the approximating Hamiltonian, the corresponding free energy density is independent of N. So is the ground state energy density, when it is obtained as a zero-temperature limit of the former. On the other hand, average values of many-particle operators over the approximating Hamiltonian may have trivial $1/N$ corrections, due to the fact that in sums of the products of spin operators, some sites coincide and others do not. Therefore, the results of the AHM make sense only in the thermodynamic limit and the $1/N$ corrections obtained within the method have no physical relevance to the original system. Let us note, as a final result, that the mean values of the considered intensive operators obtained for both the DM and LMGM coincide in the thermodynamic limit.

7. Discussion

In the last few years, a new element was introduced into the study of the DM. It was realized that the quantum phase transition in this model is relevant to the subject of quantum information and quantum computing [3, 5, 6, 10, 23, 34]. An advantage of the model is that the photon mode and the two-level atoms are natural choices for entangled subsystems. Entanglement expresses the enigmatic nonlocality inherent in quantum mechanical systems. Much effort has been devoted to elucidate the role of quantum critical phenomena (QCP) by studying quantum entanglement [11]. At a quantum critical point a modification of the ground state takes place which drastically impacts on the entanglement. It is important to understand how the entanglement depends on the order of the transition, on the range of interaction, etc. A complete theory of this phenomenon in many-particle systems is still lacking. Investigations so far have been focused on spatially one-dimensional systems or on higher-dimensional systems with special types of long-range interaction - the so called systems with infinite coordination number. The former, exactly at the point of the quantum phase transition, can be effectively described by a two dimensional Conformal Field Theory (see e.g.[11]), while the latter trivially depend on the space dimensionality and have mean-field type critical behaviour. The DM provides simple analytical solutions for different entanglement measures, at and away from the critical point λ_c, e.g. entanglement entropy, concurrence, etc [3,23,34]. The theory of DM formally represents a zero dimensional field theory which exhibits mean-field critical behaviour. From the theory of QCP, it is known that definitely pronounced quantum effects are felt long before the absolute zero temperature is reached. The revived interest in the DM is caused by the perceived relations between the thermodynamic and entanglement properties [34, 35]. It is known that mean-field models, as the ones under consideration, cannot provide nontrivial entanglement properties since the problem is effectively a single-body one in the thermodynamic limit. That is why one has to consider finite-N systems and the entanglement properties are necessarily tested in the framework of the finite-size scaling theory [34, 35].

The fact that the systems (2) and (3) are equivalent in the thermodynamic limit (on the level of free energy densities), if one focuses on the atomic degrees of freedom, has been proved [17, 20]. However, what the exact consequences of that are for the thermodynamic averages, and/or in the case of finite N, needs some further investigation, as we have pointed out. In this context, the following question arises: If two Hamiltonians generate equivalent critical behaviour as $N \to \infty$, are their finite-size properties similar? The answer= is: not always. In the case under consideration, the thermodynamic equivalence of the models (2) and (3) has been proved by the AHM. The application of this method is based on the fact that (2) and (3) have a common approximating Hamiltonian and the proof of their thermodynamic equivalence passes through the limit $N \to \infty$. For a finite N, we have just the lower and upper bounds on the difference of the free energies per spin (4). The closeness of the finite-size properties of the original and the effective model is a subtle problem, see, e.g., [31]. It has been reported (using the Holstain-Primakoff representation) that in some cases the finite-size corrections for H_D and H_N are different [34]. If this is so for thermodynamic functions, then the problem of closeness of the measures of entanglement, such as the concurrence discussed in [23, 35], is still more problematic, since these quantities probe the internal structure of the ground state in a more detailed way.

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