Variational investigation of Gaussian beam propagation in saturable nonlinear media

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Propagation of Gaussian beam in a media with saturable nonlinearity is analyzed by using variational approach. The semi-analytical variational method provides a considerable insight on the Gaussian beam dynamic. Furthermore the existence curve of such Gaussian solitons is studied numerically for which can not be evaluated exactly. Numerical results show that Gaussian function is good soliton profile in such media, and the Gaussian beam could propagate stably almost without changing their intensity shape when satisfied the existence curve, otherwise such beam propagate with a periodic or quasi-periodic manner.

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1. Introduction

It is known that propagation of beam in optical medium is an important problem in nonlinear optics. When the beam diffraction is balanced by the material nonlinearity, spatial solitons would form. Several materials are found to support solitons, such as photorefractive crystal, nonlocal media, and medium with saturable nonlinearity, etc. Different nonlinear equations are found to describe the solitons in nonlinear media. In particular, the saturable nonlinear Schrödinger equations play an important role in describing the solitons in nonlinear optics and the study of such equations have drawn considerable attention in recent years [1-18]. The soliton in several materials is described by the saturable nonlinear equation, such as non-centrosymmetric [1-6] and centro-symmetric [7-12] photorefractive media, nematic liquid crystals [13-14] etc. Even saturable square-root nonlinearity [15-16] is also possible.

There are several methods to investigate the solutions of such type of equations, for example, numerical [1-2] and exact method [3] are provided. However, there are only a few nonlinear equations with especial nonlinearity that can be solved by exact method, for example, the great success of exact analytical techniques like the inverse scattering method in solving KDV equations and nonlinear Schrödinger equations with *kerr* nonlinearity. Then much effort is made to complement the exact analytical solution methods by approximate methods. The variational method, which has been found very useful in many investigations in nonlinear optics, is a direct method based on trial functions and Rayleigh-Ritz optimization. The semi-analytical variational method provides a considerable insight on the beam dynamic. And can obtain explicit results and a clear physical picture of the properties of the solution. Quite recently, propagation of two dimensional asymmetric Gaussian beam in a medium with saturable[17] and cubic-quintic[18] absorbing nonlinear media is analyzed by using the variational approach. Gaussian solitons in nonlocal media is also investigated by variational method [19]. Inspired by their work, we investigate the propagation of a Gaussian beam in a saturable media by variational approach. And obtain the existence curve of such Gaussian solitons by evaluating the potential numerically for which can not be evaluated exactly. Furthermore numerical results show that Gaussian function is a good profile in this saturable media. The Gaussian beam could propagate stably without changing their intensity shape when satisfy the existence curve, otherwise such beam propagate with a periodic or quasi-periodic manner.

2. Theoretical model and variational analysis

To start, let us assume that the light beam propagate along the Z axis and diffract in theX direction. Propagation of an optical beam in a saturable nonlinear media is governed by the following equation [7-12],

$$i\frac{\partial A}{\partial Z}+\frac{1}{2}\frac{\partial^2 A}{\partial X^2}-\gamma(\frac{1}{1+I})^2 A=0 \tag{1}$$

Where $A(X,Z)$ is amplitude of optical field, γ is normalized nonlinear coefficient. $I(X,Z)=|A|^2$ is the normalized intensity. For small intensity, the above expression reduces to cubic-like nonlinearity; while for larger intensity, the refractive index saturates and approaches its maximum value.

Eq. (1) could be resolved by numerical method. However, in this paper we analyze the solution properties by a semi-analytical method, i.e., variational method, for which could provide clear qualitative picture and good quantitative results for the beam dynamic. Equations (1) could be written as a variational problem, and the Lagrange density is

$$L=\frac{i}{2}\left[AA_Z^*-A^*A_Z\right]+\frac{1}{2}|A_X|^2+\gamma\frac{1}{1+I} \quad (2)$$

Since we consider the propagation of Gaussian beam in this media, a Gaussian tril function is introduced, i.e.,

$$A(X,Z)=a(Z)\exp(-\frac{1}{2}\frac{X^2}{\sigma^2(Z)})\exp(i\beta(Z)X^2) \quad (3)$$

Where $a(Z)$, $\sigma(Z)$, $\beta(Z)$ are complex amplitude, beam width and curvature of the beam, respectively.

The effective Lagrange $\langle L\rangle$ is the average of L over X, i.e.,

$$\begin{aligned}\langle L\rangle&=\int_{-\infty}^{\infty}LdX\\&=\int_{-\infty}^{\infty}\left\{\frac{i}{2}\left[aa_Z^*-a^*a_Z\right]+\frac{1}{2}|a_X|^2+\gamma\frac{1}{1+I}\right\}dX\\&=\frac{\sqrt{\pi}}{2}\left\{i\sigma\left[aa_Z^*-a^*a_Z\right]+|a|^2\sigma^3\beta_Z+\frac{|a|^2(1+4\beta^2\sigma^4)}{2\sigma}+\langle L_1\rangle\right\}\end{aligned} \quad (4)$$

Where $\langle L_1\rangle=\frac{2\gamma}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{1}{1+I}dX$, $I=|a|^2\exp(-\frac{X^2}{\sigma^2})$

According to the variational principle, the following differential equations are obtained,

$$\frac{\delta\langle L\rangle}{\delta a^*}=0\Rightarrow\frac{d(i\sigma a)}{dZ}=-i\sigma a_Z+a\sigma^3\beta_Z+\frac{a(1+4\beta^2\sigma^4)}{2\sigma}+\frac{\partial\langle L_1\rangle}{\partial a^*} \quad (5)$$

$$\frac{\delta\langle L\rangle}{\delta a}=0\Rightarrow\frac{d(-i\sigma a^*)}{dZ}=i\sigma a_Z^*+a^*\sigma^3\beta_Z+\frac{a^*(1+4\beta^2\sigma^4)}{2\sigma}+\frac{\partial\langle L_1\rangle}{\partial a} \quad (6)$$

$$\frac{\delta\langle L\rangle}{\delta\sigma}=0\Rightarrow i\left[aa_Z^*-a^*a_Z\right]+3|a|^2\sigma^2\beta_Z+\frac{|a|^2(12\beta^2\sigma^4-1)}{2\sigma^2}+\frac{\partial\langle L_1\rangle}{\partial\sigma}=0 \quad (7)$$

$$\frac{\delta\langle L\rangle}{\delta\beta}=0\Rightarrow\frac{d}{dZ}(|a|^2\sigma^3)=4|a|^2\beta\sigma^3 \quad (8)$$

Above equations can be used to investigate the propagation of a Gaussian beam in saturable media.

In order to get the relation between the amplitude, beam width, we let $a^*\times(5)\pm a\times(6)$, respectively. Then after some simplification, we can obtain the following set of equations,

$$|a|^2\sigma=|a_0|^2\sigma_0=\text{costant} \quad (9)$$

$$\sigma_Z=2\beta\sigma \quad (10)$$

$$\sigma_{ZZ}=\frac{1}{\sigma^3}-\frac{1}{|a|^2\sigma}\left[\frac{a^*\partial\langle L_1\rangle/\partial a^*+a\,\partial\langle L_1\rangle/\partial a}{2\sigma}-\frac{\partial\langle L_1\rangle}{\partial\sigma}\right] \quad (11)$$

Where expression

$$a^*\partial\langle L_1\rangle/\partial a^*=a\,\partial\langle L_1\rangle/\partial a=\frac{2\gamma}{\sqrt{\pi}}|a|^2\int_{-\infty}^{\infty}\frac{\exp(-X^2/\sigma^2)}{(1+I)^2}dX \quad (12)$$

$$a\,\partial\langle L_1\rangle/\partial\sigma=\frac{2\gamma}{\sqrt{\pi}}|a|^2\int_{-\infty}^{\infty}\frac{\exp(-X^2/\sigma^2)}{(1+I)^2}\frac{2X^2}{\sigma^3}dX \quad (13)$$

Then the following results are true,

$$\sigma_{ZZ}=\frac{1}{\sigma^3}-\frac{2\gamma}{\sqrt{\pi}\sigma}\int_{-\infty}^{\infty}\frac{\exp(-X^2/\sigma^2)}{(1+I)^2}(\frac{2X^2}{\sigma^3}-\frac{1}{\sigma})dX \quad (14)$$

In generality, equation (14) is equivalent to Newtonian second law in classical mechanics for the motion of a one-dimensional particle acted by an equivalent force

$$F=\sigma_{ZZ}=\frac{1}{\sigma^3}-\frac{2\gamma}{\sqrt{\pi}\sigma}\int_{-\infty}^{\infty}\frac{\exp(-X^2/\sigma^2)}{(1+I)^2}(\frac{2X^2}{\sigma^3}-\frac{1}{\sigma})dX$$

Similar equation like Eq. (14) can be integrated once to give a potential well, and then we can analyze the dynamic of beam width. However Eq. (14) can not be evaluated exactly, we should evaluate it numerically.

If the force F is equal to zero, a soliton would form, i.e.,

$$F=\frac{1}{\sigma^3}-\frac{2\gamma}{\sqrt{\pi}\sigma}\int_{-\infty}^{\infty}\frac{\exp(-X^2/\sigma^2)}{(1+I)^2}(\frac{2X^2}{\sigma^3}-\frac{1}{\sigma})dX=0$$

. Under this condition, we can obtain

$$\gamma=\frac{\sqrt{\pi}}{2\sigma^2}\Bigg/\int_{-\infty}^{\infty}\frac{\exp(-X^2/\sigma^2)}{(1+I)^2}(\frac{2X^2}{\sigma^3}-\frac{1}{\sigma})dX \qquad (15)$$

Eq. (15) gives a relation of γ with amplitude $|a|$ when given the beam width σ.

If nonlinear coefficient γ is given, the amplitude $|a|$ could be obtained under certain beam width σ. Next we will investigate the dynamic propagation of Gaussian beam. Without loss of generality, we let $\sigma=1$.Typical numerical results are shown in Fig.1. Fig.1 (a) shows the existence curve of this Gaussian solitons, i.e., the nonlinear coefficient γ with relation to the amplitude $|a|$. Obviously amplitude bi-stability is observed in Fig.1; this means that for a given full width at half maximum (for example $\sigma=1$) there are two different values of $|a_0|$ for which bright solitary waves are to be found. Furthermore a minimum of the existence curve show that nonlinear coefficient γ should larger than this minimum value. It should be stressed that the intensity profile of the input beam is Gaussian function. And the existence curves of this "Gaussian solitons" for different beam width is like that of Fig.1. If the input beams are not Gaussian beams, the existence curve may be different.

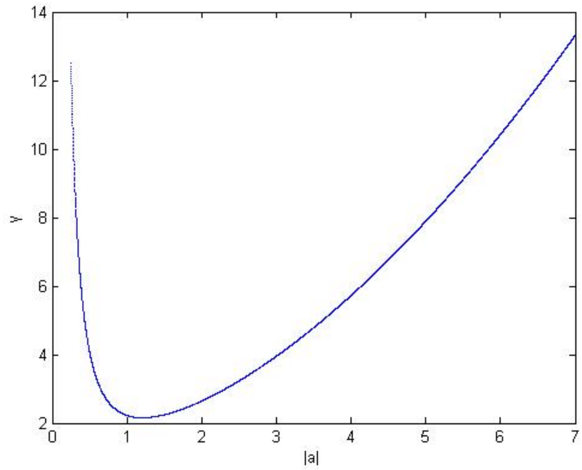
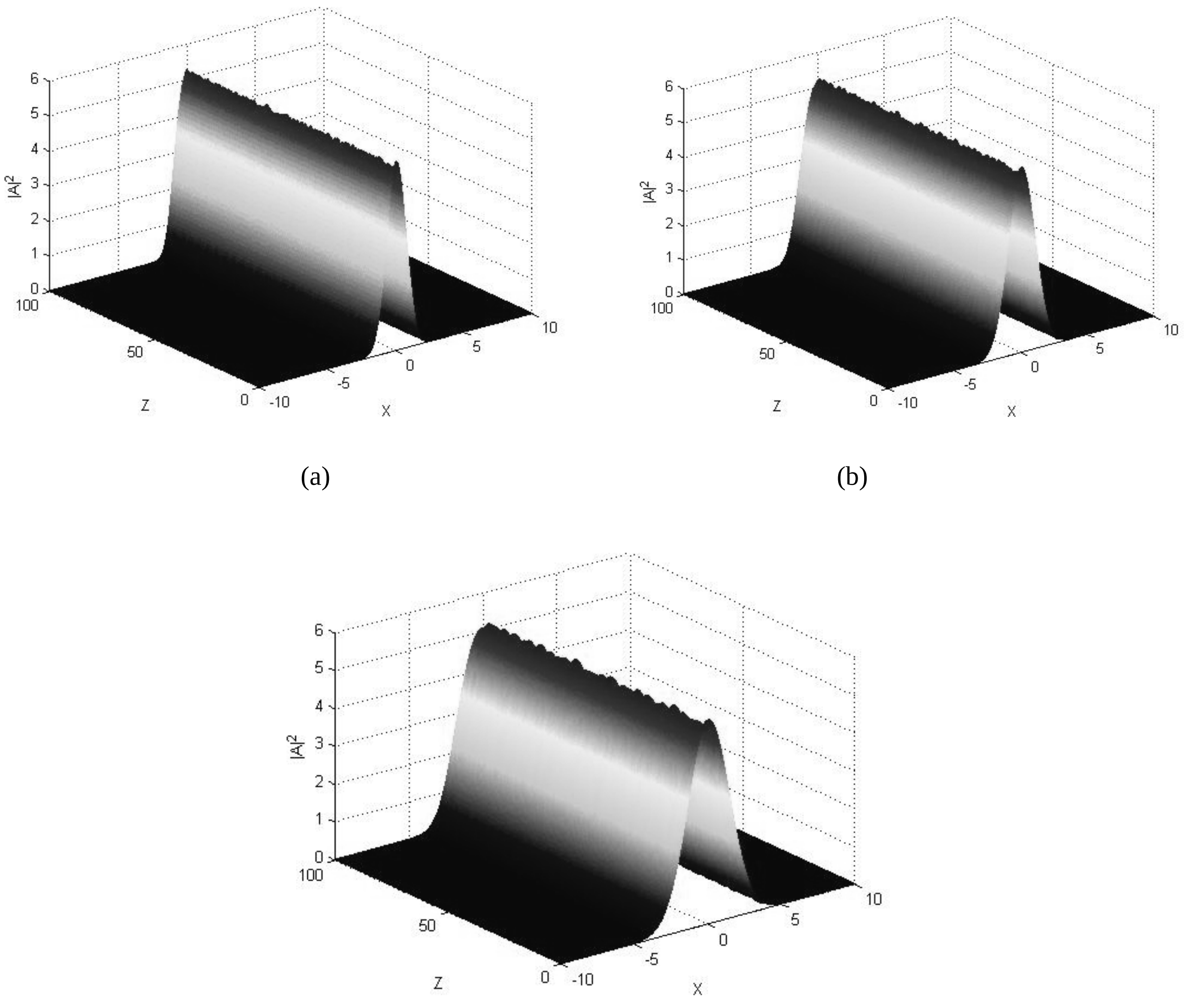


Fig. 1.Existence curve for Gaussian solitons in saturable media when $\sigma=1$.

Most important, when given the amplitude $|a_0|$, the nonlinear coefficient $\gamma(\sigma)\propto\frac{1}{\sigma^2}$. It is evident that with increasing soliton width, the bright solitary waves have a lower nonlinearity. This means larger width, lower nonlinearity. Vice verse. Typical results are shown in Fig. 2. Fig.2 (a-c) show the dynamic propagation of Gaussian beam in this media when (a) $|a_0|=2.32,\sigma_1=1,\gamma_1=3$;

(b) $|a_0|=2.32,\sigma_2=\sqrt{2},\gamma_2=1.5$;

(c) .$|a_0|=2.32,\sigma_3=2,\gamma_3=0.75$, respectively. We can see from Fig.2, the Gaussian beam propagate stably without changing their intensity shape for a distance about Z=100, and the "Gaussian solitons" is formed. Additional numerical simulations show that the Gaussian beam can propagate stably for a longer distance. All the figures in Fig.2 show that when the nonlinear coefficient $\gamma(\sigma)\propto\frac{1}{\sigma^2}$, a Gaussian soliton would form.



(a) (b)

(c)

Fig. 2 Dynamic propagation of Gaussian beam in saturable media when (a). $|a_0| = 2.32, \sigma_1 = 1, \gamma_1 = 3$ *; b) .* $|a_0| = 2.32, \sigma_2 = \sqrt{2}, \gamma_2 = 1.5$ *; (c) .* $|a_0| = 2.32, \sigma_3 = 2, \gamma_3 = 0.75$ *, respectively. Where the parameters of the input Gaussian beam satisfy the existence curve.*

If the parameters of the input Gaussian beam does not satisfy the existence curve, the Gaussian soliton can not form, and the beam will evolutes almost with a periodic or quasi-periodic manner, which can be seen from Fig. 3. Fig.3 (a-c) show the dynamic propagation of Gaussian beam in this media when (a) $|a_0| = 6, \sigma_1 = 1, \gamma_1 = 3$;

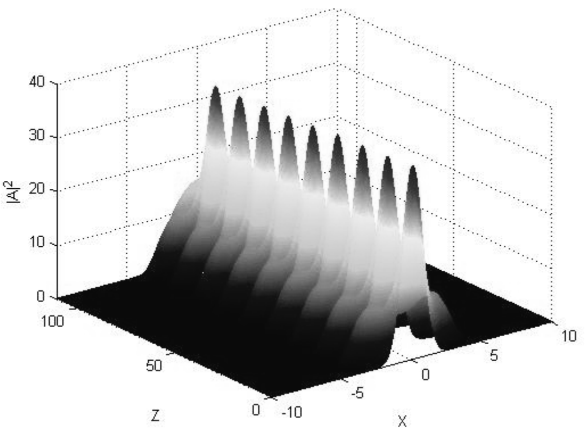
(b). $|a_0| = 5, \sigma_2 = \sqrt{2}, \gamma_2 = 1.5$;

(c). $|a_0| = 2.32, \sigma_3 = 2, \gamma_3 = 1.5$,

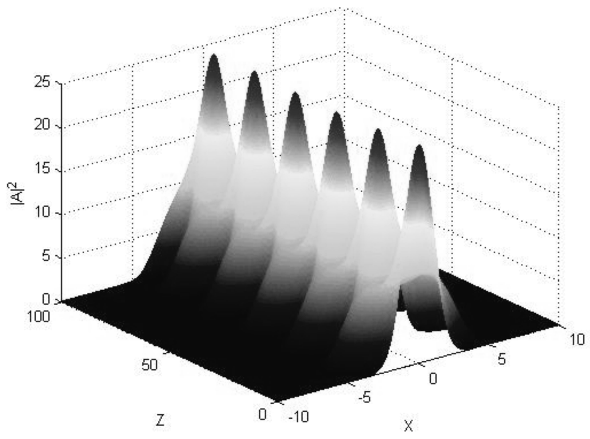
respectively. Obviously the Gaussian beam propagates almost with a periodic or quasi-periodic manner.

Comparing Fig.3 (a) with Fig. 2(a), we see that the amplitude of input beam in Fig.3 (a) is larger, so the nonlinearity cannot balance the diffraction, and then the beam diffracts at the first propagation distance, then focusing; As a result such beam propagates with a periodic or quasi-periodic manner. Comparing Fig.3 (b) with Fig. 2 (b), we can get the similar results. Finally, Comparing Fig.3 (c) with Fig. 2(c), we see that the nonlinearity in

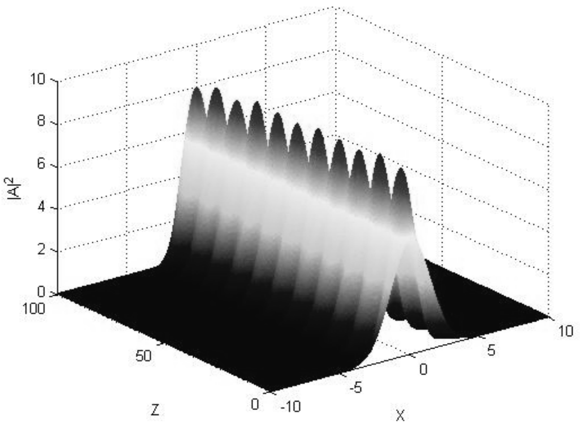
Fig.3 (c) is larger, so the diffraction could be balanced by the larger nonlinearity, so the beam focuses at the first propagation distance, then diffracting, As a result the beam propagates with a periodic or quasi-periodic manner.



(a)



(b)



(c)

Fig. 3 Dynamic propagation of Gaussian beam in saturable media when

(a) $|a_0| = 6, \sigma_1 = 1, \gamma_1 = 3$ *; (b) .* $|a_0| = 5, \sigma_2 = \sqrt{2}, \gamma_2 = 1.5$ *, (c).* $|a_0| = 2.32, \sigma_3 = 2, \gamma_3 = 1.5$ *,*

respectively. Where the parameters of the input Gaussian beam does not satisfy the existence curve.

Several issuers deserve discussion. Firstly, all the investigations confirm that Gaussian beam is a good approximation to spatial solitons in saturable media and could propagate stably when satisfy the existence curve of solitons. For the beam profile of Gaussian type is not the exact solution in such saturable media, there is some little vibration around the Gaussian solitons. Secondly, we find that the input Gaussian beam will vibrate when the parameter of Gaussian beam does not satisfy the existence curve, and the Gaussian beam propagates with a periodic or quasi-periodic manner. Thirdly, the semi-analytical variational method provides a considerable insight on the beam dynamic. And can obtain explicit results and a clear physical picture of the properties of the propagation Gaussian beams.

3. Conclusions

In conclusions, we have investigated the propagation of a Gaussian beam in saturable media by variational method, and obtained the existence curve of the Gaussian solitons by evaluating the potential numerically for which can not be evaluated exactly. Furthermore numerical results show that Gaussian function is a good profile for solitons in this saturable media. The Gaussian beam could propagate stably without changing their intensity shape when satisfy the existence curve. Otherwise such beam would propagate with a periodic or quasi-periodic manner.

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